CHAPTER

5

One-dimensional Steady State Heat Conduction with Heat Generation

5.1 Introduction

In the previous chapter, we studied one-dimensional, steady state heat conduction for a few simple geometries. In those cases, there was no internal heat generation in the medium, i.e. the term q_g appearing in the general differential equation was zero. So, the temperature distribution was determined purely by the boundary conditions. However, there are many practical cases where there is energy generation within the system and we would be interested to find out the temperature distribution within the body and the heat flux at any location, in such cases.

Examples of situations with internal heat generation are:

(i) Joule heating in an electrical conductor due to the flow of current in it
(ii) Energy generation in a nuclear fuel rod due to absorption of neutrons
(iii) Exothermic chemical reaction within a system (e.g. combustion), liberating heat at a given rate throughout the system
(iv) Heat liberated in 'shielding' used in nuclear reactors due to absorption of electromagnetic radiation such as gamma rays
(v) Curing of concrete
(vi) Magnetisation of iron
(vii) Ripening of fruits and in biological decay processes.

Temperature distribution and heat flux are of special interest in some cases where safety of the system or personnel is involved, e.g. 'burn-out' of nuclear fuel rods may occur due to excessive heat, causing a catastrophe, if suitable precautions for adequate cooling are not taken. Also, analysis of electrical machinery, transformers and electrical heaters would require that the generation of internal energy is taken into consideration.

Energy generation rate within the system is a volumetric phenomenon; so, its units are: W/m^3.

In this chapter, we shall examine the heat transfer in simple geometries (i.e. plane slabs, cylinders and spheres), with uniform internal energy generation. Several possible boundary conditions will be considered. We will study the cases with constant thermal conductivity as well as temperature dependent thermal conductivity. Finally, we will also analyse a few practical applications in the light of the theory studied with reference to these simple geometries.

5.2 Plane Slab with Uniform Internal Heat Generation

Case of a plane slab with internal heat generation has practical applications in nuclear shielding, fuel rods in nuclear reactors, electrical conductors, dielectric heating, etc.

While analysing a plane slab with internal heat generation, we shall consider three cases of boundary conditions:

(i) both the sides of the slab are at the same temperature
(ii) two sides of the slab are at different temperatures, and
(iii) one of the sides is insulated.

5.2.1 *Plane Slab with Uniform Internal Heat Generation—Both the Sides at the Same Temperature*

Consider a plane slab of thickness $2L$ as shown in Fig. 5.1. Other dimensions of the slab are comparatively large, so that heat transfer may be considered as one-dimensional in the x-direction, as shown.

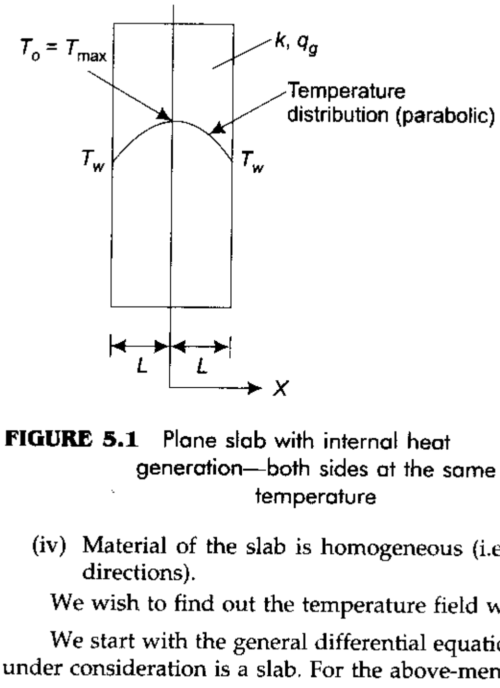

FIGURE 5.1 Plane slab with internal heat generation—both sides at the same temperature

The slab has a constant thermal conductivity k, and a uniform internal heat generation rate of q_g (W/m^3). Both the sides of the slab are maintained at the same, uniform temperature of T_w. Then, it is intuitively clear that maximum temperature will occur at the centre line, since the heat has to flow from the centre outwards. Therefore, it is advantageous to select the origin of the rectangular coordinate system on the centre line, as shown.

Let us analyse this case for temperature distribution within the slab and the heat transfer to the sides.

Assumptions:

(i) One-dimensional conduction, i.e. thickness L is small compared to the dimensions in the y and z-directions.
(ii) Steady state conduction, i.e. temperature at any point within the slab does not change with time; of course, temperatures at different points within the slab will be different.
(iii) Uniform internal heat generation rate, q_g (W/m^3).
(iv) Material of the slab is homogeneous (i.e. constant density) and isotropic (i.e. value of k is same in all directions).

We wish to find out the temperature field within the slab and then the heat flux at any point.

We start with the general differential equation in Cartesian coordinates, namely, Eq. 3.9, since the geometry under consideration is a slab. For the above-mentioned stipulations, Eq. 3.9 reduces to:

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0 \qquad ...(5.1)$$

Solution of Eq. 5.1 gives the temperature profile and then, by using Fourier's equation we get the heat flux at any point.

Two B.C.'s are required to solve this second order differential equation.

B.C.'s:

(i) a t $x = 0$, $dT/dx = 0$, since temperature is maximum at the centre line.
(ii) At $x = \pm L$, $T = T_w$

Integrating Eq. 5.1 once,

$$\frac{dT}{dx} = \frac{-q_g \cdot x}{k} + C_1 \qquad ...(a)$$

Integrating again,

$$T = \frac{-q_g \cdot x^2}{2 \cdot k} + C_1 \cdot x + C_2 \qquad ...(5.2)$$

Eq. 5.2 is the general solution for temperature distribution; this is an important equation for the slab with heat generation. Whatever may be the boundary conditions, solution is given by Eq. 5.2; only the values of integration constants C_1 and C_2 change depending on the B.C.'s.

For the present case,

applying B.C. (i) to Eq. a:

$$C_1 = 0$$

applying B.C. (ii) to Eq. 5.2:

$$T_w = \frac{-q_g \cdot L^2}{2 \cdot k} + C_2$$

i.e.

$$C_2 = T_w + \frac{q_g \cdot L^2}{2 \cdot k}$$

Substituting for C_1 and C_2 in Eq. 5.2:

$$T(x) = \frac{-q_g \cdot x^2}{2 \cdot k} + T_w + \frac{q_g \cdot L^2}{2 \cdot k}$$

i.e.

$$T(x) = T_w + \frac{+q_g}{2 \cdot k} \cdot (L^2 - x^2) \quad \text{...(5.3)}$$

where, ***L* is half-thickness of the slab.** (Remember this)

Note that the temperature, when there is internal heat generation, is not independent of k as in the case of a slab with no internal heat generation.

Also, by observation, $T = T_{max}$ at $x = 0$. (You can show this easily by differentiating Eq. 5.3 w.r.t. x and equating to zero.)

Then, putting $x = 0$ in Eq. 5.3:

$$T_{max} = T_w + \frac{q_g \cdot L^2}{2 \cdot k} \quad \text{...(5.4)}$$

Then, from Eqs. 5.3 and 5.4, we get:

$$\frac{T - T_w}{T_{max} - T_w} = \frac{L^2 - x^2}{L^2} = 1 - \left(\frac{x}{L}\right)^2 \quad \text{...(5.5)}$$

Eq. 5.5 gives the non-dimensional temperature distribution in a slab of half-thickness L, with heat generation. Note that the temperature distribution is parabolic, as shown in Fig. 5.1.

Make two important observations:

(i) From Eq. a, it is clear that temperature gradient (and, therefore, heat flux) for a slab with heat generation depends on x, whereas it was independent of x in case of a slab with no heat generation.

(ii) From Eq. 5.3, we note that temperature distribution for a slab with heat generation depends on k, whereas it was independent of k in case of a slab with no heat generation

Convection boundary condition:

In many practical applications, heat is carried away at the boundaries by a fluid at a temperature T_f flowing on the surface with a convective heat transfer coefficient, h (e.g. current carrying conductor cooled by ambient air or nuclear fuel rod cooled by a liquid metal coolant). Then, mostly, it is the fluid temperature that is known and not the wall temperature of the slab. In such cases, we relate the wall temperature and fluid temperature by an energy balance at the surface, i.e. heat conducted from within the body to the surface is equal to the heat convected away by the fluid at the surface.

In the case of a plane slab, with both sides at the same temperature, it is clear from consideration of symmetry that half the amount of heat generated travels to the surface on the right and the other half, to the left. If A is the surface area of the slab (normal to the direction of heat flow),

we have, from energy balance at the surface:

$$q_g \cdot A \cdot L = h \cdot A \cdot (T_w - T_f)$$

i.e. $$T_w = T_f + \frac{q_g \cdot L}{h} \qquad ...(5.6)$$

Substituting Eq. 5.6 in Eq. 5.3,

$$T(x) = T_f + \frac{q_g \cdot L}{h} + \frac{q_g}{2 \cdot k} \cdot (L^2 - x^2) \qquad ...(5.7)$$

Eq. 5.7 gives temperature distribution in a slab with heat generation, in terms of the fluid temperature, T_f. Remember, again, that L **is half-thickness of the slab**.

Heat transfer:

In the case of a slab with no internal heat generation, heat flux was the same at every point within the slab, since dT/dx was a constant and independent of x. However, when there is heat generation, dT/dx is not independent of x (see Eq. a), and obviously, heat flux, $q\ (= -kAdT/dx)$ varies from point to point along x. But, by observation, we know that the heat transfer rate from either of the surfaces must be equal to half of the total heat generated within the slab, for the B.C. of T_w being the same at both the surfaces.

i.e. $$Q = q_g A L \qquad ...(5.8)$$

This is easily verified by applying the Fourier's law at the surface, i.e. at $x = L$, since we now have the temperature distribution given by Eq. 5.3. We get,

$$Q = -kA(dT/dx)|_{x=L}$$

i.e. $$Q = -kA[-q_g \cdot 2x/(2k)]|_{x=L}$$

i.e. $$Q = +q_g A L \qquad \textit{(same as obtained in Eq. 5.8)}$$

5.2.1.1 Alternative analysis. In the alternative method, which is simpler, instead of starting with the general differential equation, we derive the above equations from physical considerations.

Let us consider a plane inside the slab at a distance x from the origin, as shown in Fig. 5.2.

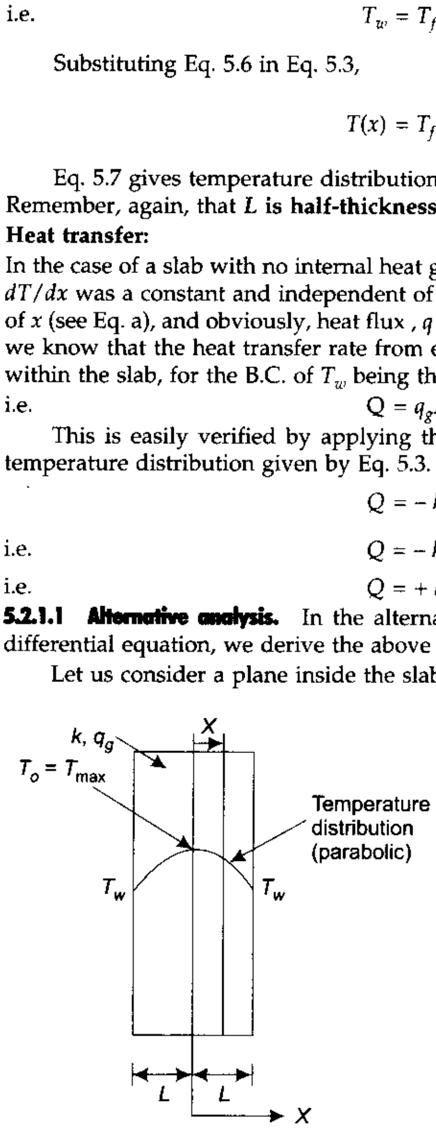

FIGURE 5.2 Plane slab with internal heat generation—both sides at the same temperature

We know from observation that maximum temperature occurs on the centre line, i.e. centre line is the line of symmetry and no heat passes across the centre line.

So, making an energy balance for the surface at a distance x from the centre line, we can write:

(Heat generated in the volume from $x = 0$ to $x = x$) = (heat leaving surface at x by conduction)

Then,

$$q_g \cdot A \cdot x = -k \cdot A \cdot \frac{dT}{dx} \qquad ...(a)$$

Separating the variables and integrating,

$$T(x) = \frac{-q_g \cdot x^2}{2 \cdot k} + C \qquad ...(b)$$

Now, at $x = 0$, $Q_x = 0$ and at $x = L$, $Q_L = q_g A L$, reaches a maximum.

At $x = L$, $T = T_w$:

Then, from Eq. b:

$$T_w = \frac{-q_g \cdot L^2}{2 \cdot k} + C$$

i.e. $$C = T_w + \frac{+q_g \cdot L^2}{2 \cdot k} \qquad ...(c)$$

Substitute C from Eq. c in Eq. *b*:

$$T(x) = T_w + \frac{q_g}{2 \cdot k} \cdot (L^2 - x^2) \quad \text{...(d) (same as Eq. 5.3)}$$

Eq. d gives the temperature distribution in the slab with heat generation, in terms of the wall temperature, T_w.

At $x = 0$, $T = T_{max}$:

Then, from Eq. d:

$$T_{max} = T_w + \frac{q_g \cdot L^2}{2 \cdot k} \quad \text{...(e)}$$

i.e.

$$T_{max} - T_w = \frac{q_g \cdot L^2}{2 \cdot k} \quad \text{...(5.9)}$$

Eq. 5.9 gives the maximum temperature difference within the slab (L is the half-thickness), when temperatures on both sides of the slab are the same. From this equation, T_{max} can be calculated, after having determined T_w from Eq. 5.6. Eq. 5.9 is, therefore, important, since in many cases, we would be interested to know the maximum temperature reached within the material, to ensure that the material will not melt in a given situation. Remember this equation.

5.2.1.2 Analysis with variable thermal conductivity. In the above analysis, thermal conductivity of the material was assumed to be constant. Now, let us make an analysis when the thermal conductivity varies linearly with temperature as:

$$k(T) = k_o(1 + \beta T),$$

where, k_o and β are constants.

Again, considering Fig. 5.2, we have from heat balance (see Eq. a above):

$$q_g \cdot x = -k(T) \cdot \frac{dT}{dx}$$

i.e.

$$-q_g \cdot x = k_o \cdot (1 + \beta \cdot T) \cdot \frac{dT}{dx}$$

Separating the variables and integrating,

$$\int (1 + \beta \cdot T) dT = \frac{-q_g}{k_o} \int x \, dx$$

i.e.

$$T + \frac{\beta \cdot T^2}{2} = \frac{-q_g}{k_o} \cdot \frac{x^2}{2} + C \quad \text{...(f)}$$

where, C is a constant, determined from the boundary condition:

Now, at $x = 0$, $T = T_o$

Then, from Eq. f,

$$C = T_o + \frac{\beta \cdot T_o^2}{2}$$

Substituting value of C in Eq. f,

$$T + \frac{\beta \cdot T^2}{2} = \frac{-q_g}{k_o} \cdot \frac{x^2}{2} + T_o + \frac{\beta \cdot T_0^2}{2}$$

i.e.

$$\frac{\beta \cdot T^2}{2} + T + \left(\frac{q_g \cdot x^2}{2 \cdot k_o} - T_o - \frac{\beta \cdot T_0^2}{2} \right) = 0 \quad \text{...(g)}$$

Eq. g is a quadratic in T. Its solution is:

$$T(x) = \frac{-1 + \sqrt{1 - 4 \cdot \frac{\beta}{2} \cdot \left(\frac{q_g \cdot x^2}{2 \cdot k_o} - T_o - \frac{\beta \cdot T_0^2}{2} \right)}}{2 \cdot \frac{\beta}{2}}$$

i.e.

$$T(x) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta^2} + \frac{2}{\beta} \cdot T_o + T_o^2 \right) - \frac{q_g \cdot x^2}{\beta \cdot k_o}}$$

i.e.

$$T(x) = \frac{-1}{\beta} + \sqrt{\left(T_o + \frac{1}{\beta} \right)^2 - \frac{q_g \cdot x^2}{\beta \cdot k_o}} \quad \text{...(5.10)}$$

Eq. 5.10 gives $T(x)$ in terms of T_o (i.e. T_{max} at $x = 0$).

Remember that x is measured from the centre line.

If we need $T(x)$ in terms of T_w, put the B.C.: at $x = L$, $T = T_w$ in Eq. f, get the value of C and then substitute C in Eq. f to get a quadratic in T. Its solution is:

$$T(x) = \frac{-1}{\beta} + \sqrt{\left(T_w + \frac{1}{\beta} \right)^2 - \frac{q_g \cdot (L^2 - x^2)}{\beta \cdot k_o}} \quad \text{..(5.11)}$$

Remember again, that L **is the half-thickness of the slab** and both the sides of the slab are maintained at the same temperature, T_w.

5.2.2 Plane Slab with Uniform Internal Heat Generation—Two Sides at Different Temperatures

Consider a plane slab of thickness L, with constant thermal conductivity k, and temperatures at the two faces being T_1 and T_2 as shown in Fig. 5.3. Coordinate system and the origin is chosen as shown.

Let $T_1 > T_2$. Now, T_{max} must occur somewhere within the slab since heat is being generated in the slab and is flowing from inside to outside, both to the left and right faces. Let T_{max} occur at a distance x_{max} from the origin, as shown in Fig. 5.3.

Our aim is to find out the temperature profile in the slab, position where the maximum temperature occurs in the slab, and the heat transfer rates to the left and right faces.

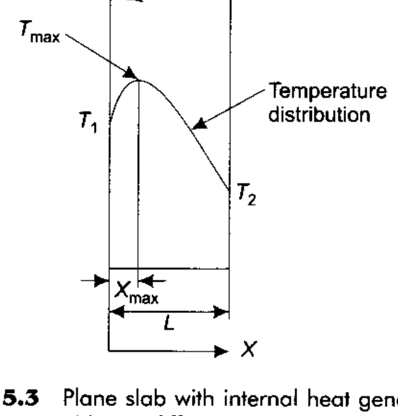

FIGURE 5.3 Plane slab with internal heat generation, two sides at different temperature

Assumptions:

(i) One-dimensional conduction, i.e. thickness L is small compared to the dimensions in the y and z directions.

(ii) Steady state conduction i.e. temperature at any point within the slab does not change with time; of course, temperatures at different points within the slab will be different.

(iii) Uniform internal heat generation rate, q_g (W/m^3).

(iv) Material of the slab is homogeneous (i.e. constant density) and isotropic (i.e. value of k is same in all directions).

Under these assumptions, as shown in section 5.2.1, the general solution for temperature distribution is given by Eq. 5.2, i.e.

$$T = \frac{-q_g \cdot x^2}{2 \cdot k} + C_1 \cdot x + C_2$$

Eq. 5.2 is the general solution for temperature distribution; C_1 and C_2 are obtained by applying the boundary conditions. For the present case, B. C.'s are:

B.C. (i): at $x = 0, T = T_1$

B.C. (ii): at $x = L, T = T_2$

Then, from B.C.(i) and Eq. 5.2, we get: $C_2 = T_1$ and, from B.C.(ii) and Eq. 5.2, we get:

$$T_2 = \frac{-q_g \cdot L^2}{2 \cdot k} + C_1 \cdot L + T_1$$

i.e.
$$C_1 = \frac{T_2 - T_1}{L} + \frac{q_g \cdot L}{2 \cdot k}$$

Substituting for C_1 and C_2 in Eq. 5.2,

$$T(x) = \frac{-q_g \cdot x^2}{2 \cdot k} + \left(\frac{T_2 - T_1}{L} + \frac{q_g \cdot L}{2 \cdot k}\right) \cdot x + T_1$$

i.e.
$$T(x) = T_1 + \left[(L - x) \cdot \frac{q_g}{2 \cdot k} + \frac{(T_2 - T_1)}{L}\right] \cdot x \qquad ...(5.12)$$

Eq. 5.12 gives the temperature distribution in the slab of thickness L, with heat generation and the two sides maintained at different temperatures of T_1 and T_2.

Location and value of maximum temperature:

To find out where the maximum temperature occurs, differentiate Eq. 5.12 w.r.t. x and equate to zero; solving, let the value of x obtained be x_{max}; substitute the obtained value of x_{max} back in Eq. 5.12 to get the value of T_{max}. This procedure will be demonstrated while solving a problem.

Heat transfer to the two sides:

Total heat generated within the slab is equal to:

$$Q_{tot} = q_g A L$$

Part of this heat moves to the left and gets dissipated at the left face; remaining portion of the heat generated moves to the right and gets dissipated from the right face.

Applying Fourier's law:

$$Q_{right} = -k\,A\,(dT/dx)|_{x=L}$$

$$Q_{left} = -k\,A\,(dT/dx)|_{x=0}$$

(this will be negative since heat flows from right to left, i.e. in negative x-direction)

Of course, sum of Q_{right} and Q_{left} must be equal to Q_{tot}.

Convection boundary condition:

Let heat be carried away at the left face by a fluid at a temperature T_a flowing on the surface with a convective heat transfer coefficient, h_a, and on the right face, by a fluid at a temperature T_b flowing on the surface with a convective heat transfer coefficient, h_b. In such cases, we relate the wall temperature and fluid temperature by an energy balance at the surfaces, i.e. heat conducted from within the body to the surface is equal to the heat convected away by the fluid at the surface.

Further, the maximum temperature occurs at $x = x_{max}$, already calculated. Then, heat generated in the slab in the volume between $x = 0$ and $x = x_{max}$ has to move to the left face and the heat generated in the volume between $x = x_{max}$ and $x = L$ has to move to the right face, since no heat can cross the plane of maximum temperature.

Then, we have, from energy balance at the two surfaces:

On the left face:

$$q_g \cdot A \cdot x_{max} = h_a \cdot A \cdot (T_1 - T_a) \qquad ...(a)$$

On the right face:

$$q_g \cdot A \cdot (L - x_{max}) = h_b \cdot A \cdot (T_2 - T_b) \qquad ...(a)$$

From Eqs. a and b, we get T_1 and T_2 in terms of known fluid temperatures T_a and T_b, respectively. Thus after obtaining T_1 and T_2, substitute them in Eq. 5.12 to get the temperature distribution in terms of fluid temperatures T_a and T_b.

5.2.2.1 Effect of variable k, for a slab with heat generation, and two sides at different temperatures. For the assumptions of one-dimensional, steady state conduction with uniform heat generation and k varying with temperature linearly as: $k(T) = k_o(1 + \beta T)$, the controlling differential equation is (see Chapter 3):

$$\frac{d}{dx}\left(k(T)\cdot\frac{dT}{dx}\right) + q_g = 0$$

Integrating $\quad k(T)\cdot\frac{dT}{dx} + q_g\cdot x = C_1$ where C_1 is a constant.

Separating the variables and integrating:

$$\int k(T)\,dT = \int (C_1 - q_g\cdot x)\,dx$$

Substituting for $k(T)$: $\int [k_o\cdot(1 + \beta\cdot T)]\,dT = \int (C_1 - q_g\cdot x)\,dx$

i.e. $$T + \frac{\beta\cdot T^2}{2} = \frac{1}{k_o}\cdot\left(C_1\cdot x - \frac{q_g\cdot x^2}{2} + C_2\right)$$ *((A)...where C_1 and C_2 is a constant of integration)*

C_1 and C_2 are found out by applying the B.C.'s in Eq. A:

B.C.(i): at $x = 0$, $T = T_1$

B.C.(ii): at $x = L$, $T = T_2$

From B.C.(i) and Eq. A, we get:

$$C_2 = k_o\cdot\left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)$$

From B.C.(ii) and Eq. A, we get:

$$T_2 + \frac{\beta\cdot T_2^2}{2} = \frac{1}{k_o}\cdot\left[C_1\cdot L - \frac{q_g\cdot L^2}{2} + k_o\cdot\left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)\right]$$

Therefore, $$C_1 = \frac{k_o}{L}\cdot\left(T_2 + \frac{\beta\cdot T_2^2}{2}\right) + \frac{q_g\cdot L}{2} - \frac{k_o}{L}\cdot\left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)$$

Substituting values of C_1 and C_2 in Eq. A:

i.e. $$T + \frac{\beta\cdot T^2}{2} = \frac{x}{L}\cdot\left(T_2 + \frac{\beta\cdot T_2^2}{2}\right) + \frac{q_g\cdot L\cdot x}{2\cdot k_o} - \frac{x}{L}\cdot\left(T_1 + \frac{\beta\cdot T_1^2}{2}\right) - \frac{q_g\cdot x^2}{2\cdot k_o} + \left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)$$

i.e. $$T + \frac{\beta\cdot T^2}{2} = \left(T_2 + \frac{\beta\cdot T_2^2}{2}\right)\cdot\frac{x}{L} + \frac{q_g\cdot x}{2\cdot k_o}\cdot(L - x) + \left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)\cdot\left(1 - \frac{x}{L}\right)$$

i.e. $$\beta\cdot\frac{T^2}{2} + T - \left[\left(T_2 + \frac{\beta\cdot T_2^2}{2}\right)\cdot\frac{x}{L} + \frac{q_g\cdot x}{2\cdot k_o}\cdot(L - x) + \left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)\cdot\left(1 - \frac{x}{L}\right)\right] = 0$$

This is a quadratic in T. Its positive root is:

$$T(x) = \frac{-1 + \sqrt{1 + 4\cdot\frac{\beta}{2}\cdot\left[\left(T_2 + \frac{\beta\cdot T_2^2}{2}\right)\cdot\frac{x}{L} + \frac{q_g\cdot x}{2\cdot k_o}\cdot(L - x) + \left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)\cdot\left(1 - \frac{x}{L}\right)\right]}}{2\cdot\frac{\beta}{2}}$$

i.e. $$T(x) = \frac{-1}{\beta} + \sqrt{\frac{1}{\beta^2} + \frac{2}{\beta}\cdot\left[\left(T_1 + \frac{\beta\cdot T_1^2}{2}\right) - \frac{x}{L}\cdot\left(T_1 + \frac{\beta\cdot T_1^2}{2} - T_2 - \frac{\beta\cdot T_2^2}{2}\right) + \frac{q_g\cdot x}{2\cdot k_o}\cdot(L-x)\right]}$$

i.e. $$T(x) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta^2} + T_1^2 + \frac{2\cdot T_1}{\beta}\right) - \frac{2\cdot x}{\beta\cdot L}\cdot\left[(T_1 - T_2) + \frac{\beta}{2}\cdot(T_1 - T_2)\cdot(T_1 + T_2)\right] + \frac{q_g\cdot x}{\beta\cdot k_o}\cdot(L-x)}$$

i.e. $$T(x) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_1\right)^2 - \frac{2\cdot x}{\beta\cdot L}\cdot(T_1 - T_2)\cdot(1 + \beta\cdot T_m) + \frac{q_g\cdot x}{\beta\cdot k_o}\cdot(L-x)}$$

where, $T_m = \dfrac{T_1 + T_2}{2}$ *(mean temperature.)*

Eq. 5.12a gives the temperature distribution in a slab of thickness L, with heat generation, with the two faces maintained at different temperatures, when the k varies linearly with temperature.

5.2.3 Plane Slab with Uniform Internal Heat Generation— One Face Perfectly Insulated

Consider a plane slab of thickness L, with constant thermal conductivity k, and one of the faces (say, left face) is insulated as shown in Fig. 5.4. Other face of the slab is at a temperature of T_w. Coordinate system and the origin is chosen as shown.

Now, T_{max} must occur on the insulated left surface of the slab since heat is being generated in the slab and is constrained to flow from left face to right face.

Our aim is to find out the temperature profile in the slab, and the heat transfer rate.

Assumptions:

(i) One-dimensional conduction, i.e. thickness L is small compared to the dimensions in the y and z directions.

(ii) Steady state conduction, i.e. temperature at any point within the slab does not change with time; of course, temperatures at different points within the slab will be different.

(iii) Uniform internal heat generation rate, q_g (W/m^3).

(iv) Material of the slab is homogeneous (i.e. constant density) and isotropic (i.e. value of k is same in all directions).

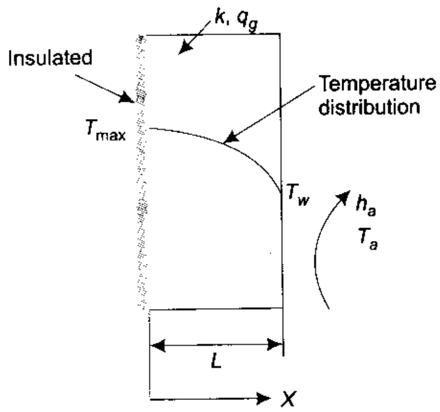

FIGURE 5.4 Plane slab with internal heat generation, one side insulated

Under these assumptions, as shown in section 5.2.1, the general solution for temperature distribution is given by Eq. 5.2, i.e.

$$T = \frac{-q_g\cdot x^2}{2\cdot k} + C_1\cdot x + C_2 \quad \text{...(5.2)}$$

Eq. 5.2 is the general solution for temperature distribution; C_1 and C_2 are obtained by applying the boundary conditions. For the present case, B. C.'s are:

B.C.(i): at $x = 0$, $dT/dx = 0$, since perfectly insulated. (Note: 'perfectly insulated' means that $Q = 0$, i.e. $-k\,A\,(dT/dx) = 0$, and since k and A are not zero, dT/dx must be zero).

B.C.(ii): at $x = L$, $T = T_w$

From Eq. 5.2:

$$\frac{dT}{dx} = \frac{-q_g\cdot x}{k} + C_1$$

Then applying B.C.(i), we get: $C_1 = 0$
From B.C.(ii) and Eq. 5.2:

$$C_2 = T_w + \frac{q_g \cdot L^2}{2 \cdot k}$$

Substituting for C_1 and C_2 in Eq. 5.2:

$$T(x) = T_w + \frac{q_g}{2 \cdot k} \cdot (L^2 - x^2) \quad ...(5.13)$$

Eq. 5.13 gives the temperature distribution in a slab of thickness L, with heat generation when one side is perfectly insulated. Fig. 5.4 shows the temperature distribution in the slab; note that temperature curve should approach the left face horizontally, since $(dT/dx) = 0$ at $x = 0$.

Note that Eq. 5.13 is the same as Eq. 5.3, which was derived for a slab with heat generation, when both the sides were maintained at the same temperature, except that now, **L is the thickness of the slab** and not half-thickness.

In case of convection boundary condition:

Let the heat be lost from the un-insulated surface to a fluid at T_a, flowing on the surface with a heat transfer coefficient of h_a. Then, we relate T_w and T_a by making an energy balance at the right face. Since the left face is insulated, all the heat generated in the slab travels to the surface on the right and gets convected away to the fluid.

Heat generated in the slab:

$$Q_{gen} = q_g \cdot A \cdot L$$

Heat convected at surface:

$$Q_{conv} = h_a \cdot A \cdot (T_w - T_a)$$

Equating the heat generated and heat convected, we get:

$$T_w = T_a + \frac{q_g \cdot L}{h_a} \quad ...(a)$$

Substituting from (a) in Eq. 5.13,

$$T(x) = T_a + \frac{q_g \cdot L}{h_a} + \frac{q_g}{2 \cdot k} \cdot (L^2 - x^2) \quad ...(5.14)$$

Eq. 5.14 gives the temperature distribution in a slab with heat generation and constant k, insulated at one face and losing heat at the other face to a fluid by convection, in terms of the fluid temperature.

Note: If the convection resistance is zero, which means that the heat transfer coefficient is infinity, the wall temperature and the fluid temperature are the same, i.e. $T_a = T_w$, and Eq. 5.14 reduces to Eq. 5.13.

Maximum temperature:

Obviously, maximum temperature occurs at the insulated surface. This can be easily verified by differentiating the expression for temperature distribution, Eq. 5.13, w.r.t. x and equating to zero. Putting $x = 0$ in Eqn. 5.13:

$$T_{max} = T_w + \frac{q_g \cdot L^2}{2 \cdot k} \quad ...(5.15)$$

Eq. 5.15 gives T_{max} in terms of wall temperature, T_w.
Substituting for T_w from Eq. a in Eq. 5.15:

$$T_{max} = T_a + \frac{q_g \cdot L}{h_a} + \frac{q_g \cdot L^2}{2 \cdot k} \quad ...(5.16)$$

Eq. 5.16 gives T_{max} in terms of fluid temperature, T_a.
From Eq. 5.13 and 5.15, we can write:

$$\frac{T(x) - T_w}{T_{max} - T_w} = \frac{L^2 - x^2}{L^2} = 1 - \left(\frac{x}{L}\right)^2 \quad ...(5.17)$$

Eqn. 5.17 gives non-dimensional temperature distribution for a slab with heat generation, and one face insulated. This equation is the same as Eq. 5.5 for a slab with heat generation and both faces at the same temperature, except that, now L is the thickness of the slab (In the case of Eq. 5.5, L was the half-thickness).

Example 5.1. Heat is generated uniformly in a stainless steel plate having $k = 20$ W/(mK). The thickness of the plate is 1 cm and heat generation rate is 500 MW/m^3. If the two sides of the plate are maintained at 100°C and 200°C, respectively, calculate:

(i) the temperature at the centre of the plate
(ii) the position and value of maximum temperature
(iii) heat transfer at the left and right faces
(iv) sketch the temperature profile in the slab.

Solution.

Data:

$L := 0.01$ m $\quad A := 1$ m^2 $\quad k := 20$ W/(mc) $\quad q_g := 500 \times 10^6$ W/m^3 $\quad T_1 := 200$ C $\quad T_2 := 100$ C

See Fig. Example 5.1.

This is the case of one-dimensional, steady state conduction through a plate with heat generation, when the two faces of the plate are maintained at different temperatures. So, we can directly apply Eq. 5.12 to get $T(x)$ at any position x.

However, let us solve this problem from first principles, and then verify the result from Eq. 5.12.

For this situation, governing differential eqution is:

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

Integrating: $$\frac{dT}{dx} + \frac{q_g \cdot x}{k} = C_1 \qquad \text{...(b)}$$

Integrating again: $$T(x) = \frac{-q_g \cdot x^2}{2 \cdot k} + C_1 \cdot x + C_2 \qquad \text{...(c)}$$

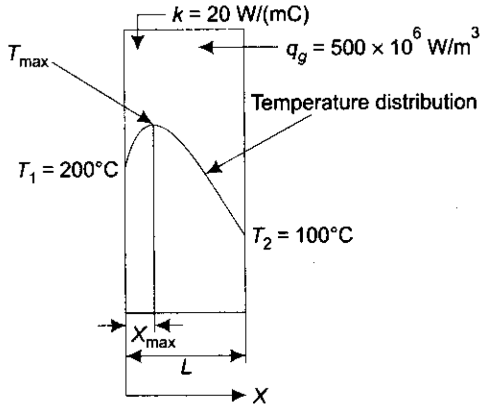

FIGURE Example 5.1 Plane slab with internal heat generation, two sides at different temperature

Apply the B.C.'s: i.e.

(i) at $x = 0$: $\quad T_1 = 200°C$

(ii) at $x = L = 0.01$ m: $\quad T_2 = 100°C$

From B.C. (i) and Eq. c: $C_2 = 200$

From B.C. (ii) and Eq. c:

$$T_2 = \frac{-q_g \cdot L^2}{2 \cdot k} + C_1 \cdot L + C_2$$

i.e. $$C_1 := \frac{\left(T_2 + \frac{q_g \cdot L^2}{2 \cdot k} - C_2\right)}{L} \qquad \text{...define } C_1$$

i.e. $$C_1 = 1.15 \times 10^5$$

Substituting for C_1 and C_2 in Eq. c:

$$T(x) := \frac{-500 \times 10^6 \cdot x^2}{2 \times 20} + 1.15 \times 10^5 \cdot x + 200 \qquad \text{((d) (define } T(x)))$$

Eq. d gives the temperature profile.

Temperature at the centre of the plate:

Put $x = 0.005$ m in Eq. d:

i.e. $\quad T(0.005) = 462.5°C \qquad$ ***(temperature at the centre of plate.)***

Verify: from direct formula Eq. 5.12:

$$T(x) := T_1 + \left[(L - x) \cdot \frac{q_g}{2 \cdot k} + \frac{T_2 - T_1}{L}\right] x \qquad \text{...(5.12)}$$

Put $x = 0.005$ m: $\quad T(0.005) = 462.5$ °C $\qquad$ ***(verified.)***

Position and value of maximum temperature:

We have the relation for $T(x)$ in Eq. d. Differentiate it w.r.t. x and equate to zero. Root of the resulting equation gives the position, x_{max} of the location of maximum temperature. Then, substitute x_{max} back in Eq. d to get T_{max}.

We have:

$$T(x) := \frac{-500 \times 10^6 \cdot x^2}{2 \cdot 20} + 1.15 \times 10^5 \cdot x + 200 \qquad \text{(Eq. d...define } T(x))$$

Let

$$T'(x) = \left(\frac{d}{dx} T(x)\right)$$

Then,

$$T'(x) = \frac{-500 \times 10^6 \cdot 2 \cdot x}{2 \cdot 20} + 1.15 \times 10^5 \qquad \text{...(d)}$$

Putting $T'(x) = 0$ and solving:

$$x = \frac{1.15 \times 10^5 \times 2 \times 20}{500 \times 10^6 \times 2}$$

i.e. $x = 4.6 \times 10^{-3}$ m = 4.6 mm = x_{max}...***position of maximum temperature from LHS.***

i.e. $x_{max} = 0.0046$ m.

Verify: In Mathcad, there is no need to do the labour of differentiation, equating to zero and then solving for x, as done above.

Instead, define $T'(x)$ as the first derivative of $T(x)$ w.r.t. x and use the 'root function' to find the root of $T'(x) = 0$: For this, first, assume a trial (guess) value of x:

$$T'(x) := \left(\frac{d}{dx} T(x)\right) \qquad \textit{(define } T'(x))$$

$$x := 0.002 \qquad \textit{(trial value of } x)$$

$$x_{max} := \text{root}\,(T'(x), x) \qquad \text{(define } x_{max} \text{ as the root of equation } T'(x) = 0)$$

i.e. $x_{max} = 4.6 \times 10^{-3}$ m ***(position of maximum temperature from LHS...verified.)***

Value of maximum temperature:

This is obtained by putting the value of x_{max} in Eq. d:

i.e. put $x = x_{max}$ in T(x):

$$T_{max} := T(0.0046)\ °C \qquad \textit{(define } T_{max})$$

i.e. $T_{max} = 464.5°C$ ***(value of maximum temperature.)***

Heat transfer to left and right faces:

Knowing $T(x)$, it is easy to find $T'(x) = (dT/dx)$ at $x = 0$ and at $x = L$; We have already found out, in Eq. d, $T'(x)$ – just put $x = 0$ or $x = L$, as required. Then, apply Fourier's law to get Q at $x = 0$ and $x = L$:

Heat transfer from left face, Q_1:

$$Q_1 := -k \cdot A \cdot T'(0) \qquad \textit{(define } Q_1 \textit{...Fourier's law)}$$

i.e. $Q_1 = -2.3 \times 10^6$ W/m^2 = 2300 kW/m^2 ...***heat transfer from left face.***

Note that negative sign indicates that heat flow is in a direction opposite to the positive X-direction, i.e. heat flow is from right to left, as far as the left face is concerned. Heat is flowing from centre to left side in steady state.

Heat transfer from right face, Q_2:

$$Q_2 := -k \cdot A \cdot T'(0.01)$$

i.e. $Q_2 = 2.7 \times 10^6$ W/m^2 = 2700 kW/m^2 ***(heat transfer from right face.)***

Verify: Sum of Q_1 and Q_2 must be equal to the total heat generated, Q_{gen}

$$Q_{gen} := q_g \cdot A \cdot L,\ W \qquad \textit{(define total heat generated)}$$

i.e. $Q_{gen} = 5 \times 10^6$, W *(total heat generated)*

Also, $|Q_1| + |Q_2| = 5 \times 10^6$, W = Q_{gen} ***(verified)***

Note: Remember that absolute values of Q_1 and Q_2 are to be used, disregarding the signs, since the sign only indicates the direction of heat flow.

To plot the temperature profile in the slab:

This is done very easily in Mathcad. First, define a range variable x, varying from 0 to 0.01 m, with an increment of 0.0005 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.1(b).

$x := 0, 0.0005, ..., 0.01$ *(define a range variable x starting value = 0, next value = 0.0005 m and last value = 0.01 m)*

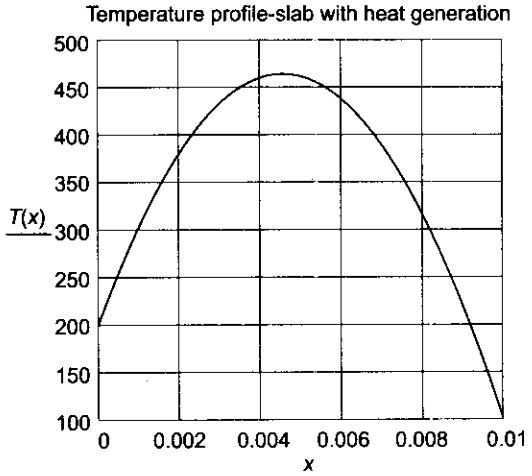

FIGURE Example 5.1(b)

Note: It may be observed from the graph that the maximum temperature is 464.5°C and it occurs at $x = 0.0046$ m.

Example 5.2. If in Example 5.1, the temperatures on either side of the plate are maintained at 100°C, calculate:

(i) the temperature on the centre line

(ii) temperature at one-quarter of the thickness from the surface

(iii) draw the temperature profile.

Solution.

Data:

$2L = 0.01$ m $\quad L := 0.005$ m $\quad A := 1\ \text{m}^2$

$k := 20$ W/(m.c) $\quad q_g := 500 \times 10^6\ \text{W/m}^3\ T_w := 100°\text{C}$

See Fig. Example 5.2.

This is the case of one-dimensional, steady state conduction through a plate with heat generation, when the two faces of the plate are maintained at the same temperature. So, we can directly apply Eq. 5.3 to get $T(x)$ at any position x.

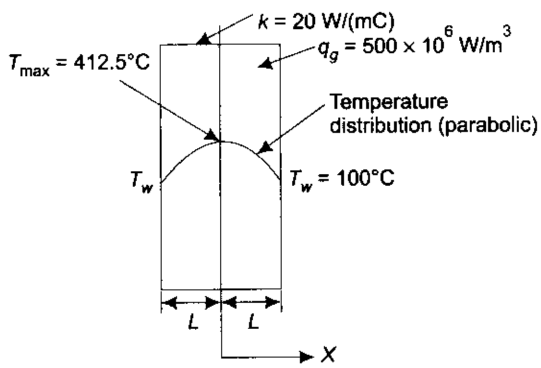

FIGURE Example 5.2 Plane slab with internal heat generation, both sides at the same temperature

i.e.
$$T(x) := T_w + \frac{q_g}{2 \cdot k} \cdot (L^2 - x^2) \qquad \text{...(5.3)}$$

where, L is half-thickness of the slab.

Temperature at the centre line of plate:

At mid-plane, $x = 0$; therefore, substitute $x = 0$ in Eq. 5.3:

$T(0) = 412.5°\text{C}$ ***(temperatue at the centre line of plate.)***

$T(0) = 412.5°\text{C}$ is also the **maximum temperature in the plate.**

Temperature at one-quarter the thickness from the surface:

i.e. at $x = 0.00025$ m from the centre line. Put $x = 0.00025$ in Eq. 5.3:

$T(0.00025) = 334.375°\text{C}$ ***(temperature at 1/4 of the thickness from surface.)***

To draw the temperature profile:

We shall draw the temperature profile for the right half; by symmetry, temperature profile on the left half is the mirror image of that on the right half.

First, define a range variable x, varying from 0 to 0.005 m, with an increment of 0.00025 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.2(b).

$x := 0, 0.00025, \ldots, 0.005$ *(define a range variable x..starting value = 0, next value = 0.00025 m and last value 0.0005 m)*

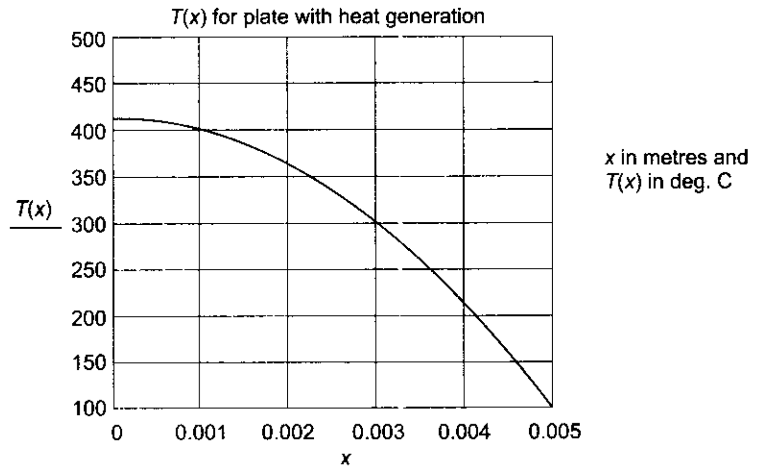

FIGURE Example 5.2(b)

Note: Above graph shows the temperature profile for the right half of a plate with internal heat generation, when both the sides are maintained at 100°C. For the left side of the plate, temperature profile is identical, mirror image of this graph.

Example 5.3. In Example 5.1, if the thermal conductivity of the material varies as: $k(T) = k_o(1 + \beta T)$, (W/(mC) where k_o = 14.695 W/(mC) and $\beta = 10.208 \times 10^{-4}$, (C^{-1}), and T is in deg.C.

(i) calulate the temperature on the centre line
(ii) find location and value of maximum temperature in the plate
(iii) find heat transfer rate to the left and right sides, and
(iv) draw the temperature profile.

Solution. See Fig. Example 5.3.

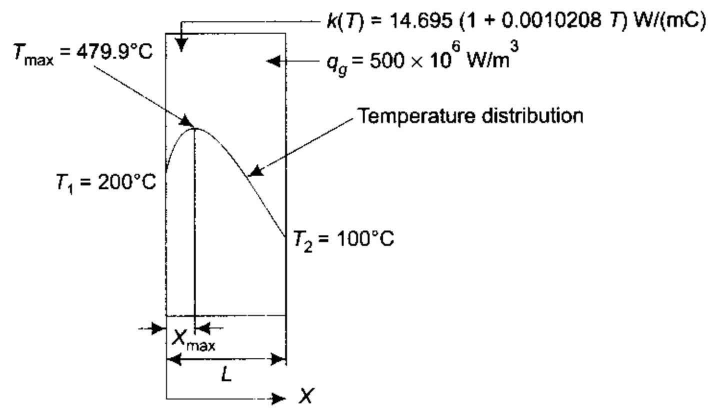

FIGURE Example 5.3 Plane slab with internal heat generation, variable k, with two sides at different temperature

Data:

$L := 0.01$ m $\quad A := 1$ m^2 $\quad k(T) := k_o\cdot(1 + \beta\cdot T)$ $\quad k_o := 14.965$ W/(mC) $\quad \beta := 10.208 \times 10^{-4}$ (1/C)

$q_g := 500 \times 10^6$ W/m^3 $\quad T_1 := 200°C$ $\quad T_2 := 100°C$

We can directly use Eq. 5.13 to get the temperature at any location; however, let us work out this problem from fundamentals and verify the result from Eq. 5.13.

We start with the governing differential equation for the case of a slab in steady state, one dimensional conduction with heat generation and variable k, and integrate it twice in conjunction with the B'C.'s, to get the temperature profile:

We have:

$$\frac{d}{dx}\left(k(T)\cdot\frac{dT}{dx}\right) + q_g = 0$$

Integrating: $k(T)\cdot\frac{dT}{dx} + q_g\cdot x = C_1$ where C_1 is a constant.

Separating the variables and integrating:

$$\int k(T)dT = \int (C_1 - q_g\cdot x)dx$$

Substituting for $k(T)$: $\int [k_o\cdot(1 + \beta\cdot T)]dT = \int (C_1 - q_g\cdot x)dx$

i.e.

$$T + \frac{\beta\cdot T^2}{2} = \frac{1}{k_o}\cdot\left(C_1\cdot x - \frac{q_g\cdot x^2}{2} + C_2\right) \quad \text{...(a)}$$

Eq. a is the general equation for temperature distribution. Constants C_1 and C_2 are obtained by applying the B.C.'s:

B.C.(i): at $x = 0$, $T = T_1$

B.C.(ii): at $x = L$, $T = T_2$

From B.C.(i) and Eq. a $\quad C_2 := k_o\cdot\left(T_1 + \frac{\beta\cdot T_1^2}{2}\right)$

substituting $\quad C_2 = 3.239 \times 10^3$

From B.C.(ii) and Eq. a

$$C_1 := \frac{k_o\cdot\left(T_2 + \frac{\beta\cdot T_2^2}{2}\right) + \frac{q_g\cdot L^2}{2} - C_2}{L}$$

substituting, we get $\quad C_1 = 2.331 \times 10^6$.

Substituting value of C_1 and C_2 in Eq. a and simplifying, we get:

$$\frac{\beta\cdot T^2}{2} + T - (1.58595 \times 10^5\cdot x - 1.70126 \times 10^7\cdot x^2 + 220.416) = 0 \quad \text{...(b)}$$

Eq. b is a quadratic in T. Its positive root is given by:

$$T(x) := \frac{-1 + \sqrt{1 + 4\cdot\frac{10.208\times10^{-4}}{2}\cdot(1.58595\times10^5\cdot x - 1.70126\times10^7\cdot x^2 + 220.416)}}{2\cdot\frac{10.208\times10^{-4}}{2}} \quad \text{...(c)}$$

Eq. c gives the variation of temperature with x.

Temperature at the centre line:

Put $x = 0.005$ m in Eq. c

$T(0.005) = 473.597°C$ ***(temperature at centre line.)***

Verify: verify this result from direct formula, eqn. 5.12, a.

$T_m = \frac{T_1 + T_2}{2}$ i.e. $T_m = 150°C$ *(mean value of temperature)* i.e. $$T(x) := \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta}+T_1\right)^2 - \frac{2\cdot x}{\beta\cdot L}\cdot(T_1-T_2)\cdot(1+\beta\cdot T_m) + \frac{q_g\cdot x}{\beta\cdot k_o}\cdot(L-x)} \quad ...5.12(a)$$

i.e. $T(0.005) = 473.596°C$ *(verified.)*

Location and value of maximum temperature in the plate:

Differentiate Eq. c w.r.t. x and equate to zero and get x_{max}, the position of T_{max}; substitute this value of x_{max} back in Eq. c to get value of T_{max}.

In Mathcad, we do not have to go through the labour of differentiation, equating to zero, then solving etc. We use the 'root function'. First, define $T'(x) = d(T(x))/dx$. Then, assume a trial value of x and type the command 'root($T'(x)$, $x$trial) = '. This gives root of $T'(x) = 0$.

$$T(x) := \frac{-1+\sqrt{1+4\cdot\frac{10.208\times10^{-4}}{2}\cdot(1.58595\times10^{5}\cdot x - 1.70126\times10^{7}\cdot x^2 + 220.416)}}{2\cdot\frac{10.208\times10^{-4}}{2}} \quad ...define\ T(x)...(c)$$

$$T'(x) := \frac{d}{dx}T(x) \quad ...define\ T'(x)$$

$x := 0.002$ m *(trial value of x)*

$x_{max} :=$ root $(T'(x), x)$ *(define x_{max})*

i.e. $x_{max} = 4.661 \times 10^{-3}$ m = 4.661 mm ***(location of maximum temperature...distance from LHS.)***

Value of maximum temperature is obtained by putting $x = x_{max}$ in Eq. c

i.e. $T(x_{max}) = 474.913°C$ ***(value of maximum temperature)***

Heat transfer to left and right faces:

Knowing $T(x)$, it is easy to find $T'(x) = (dT/dx)$ at $x = 0$ and at $x = L$; We have already defined $T'(x)$—just put $x = 0$ or $x = L$, as required. Then, apply Fourier's law to get Q at $x = 0$ and $x = L$:

Remember $k(T) := k_o\cdot(1 + \beta\cdot T)$ *(define k(T))*

Heat transfer from left face, Q_1:

$Q_1 := -k(T_1)\cdot A\cdot T'(0)$ *(define Q_1...Fourier's law)*

i.e. $Q_1 = -2.331 \times 10^6$ W/m^2 = – 2331 kW/m^2 ***(heat transfer from left face)***

Note: Negative sign indicates that heat is flowing from right to left, i.e. in the negative X-direction.

Check: This should equal the amount of heat generated between $x = 0$ and $x = x_{max}$

Heat generated between $x = 0$ and $x = x_{max}$:

$Q_{gen1} := q_g\cdot A\cdot(x_{max} - 0)$

i.e. $Q_{gen1} = 2.331 \times 10^6$ W/m^2 *(verified.)*

Heat transfer from right face, Q_2:

$Q_2 := -k(T_2)\cdot A\cdot T'(0.001)$ *(define Q_2...Fourier's law)*

i.e. $Q_2 = 2.669 \times 10^6$ W/m^2 = 2669 kW/m^2 ***(heat transfer from left face)***

Check: This should equal the amount of heat generated between $x = x_{max}$ and $x = L$.

Heat generated between $x = 0$ and $x = x_{max}$ and $x = L$

$Q_{gen2} := q_g\cdot A\cdot(L - x_{max})$

i.e. $Q_{gen2} = 2.669 \times 106$ W/m^2 *(verified.)*

To plot the temperature profile in the plate:

This is done very easily in Mathcad. First, define a range variable x, varying from 0 to 0.01 m, with an increment of 0.0005 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.3(b).

$x := 0, 0.0005, ..., 0.01$ *(define a range variable x..starting value = 0, next value = 0.0005 m and last value = 0.01 m)*

Note: It may be observed from the graph that the maximum temperature is 474.9°C and it occurs at $x = 0.00466$ m.

Example 5.4. A plane wall of thickness 0.1 m and k = 25 W/(mK), having uniform volumetric heat generation of 0.3 MW/m^3 is insulated on one side and is exposed to a fluid at 92°C. The convective heat transfer coefficient between the wall and the fluid is 500 W/(m^{2}K). Determine:

(i) the maximum temperature in the wall
(ii) temperature at the surface exposed to the fluid
(iii) Draw the temperature profile.

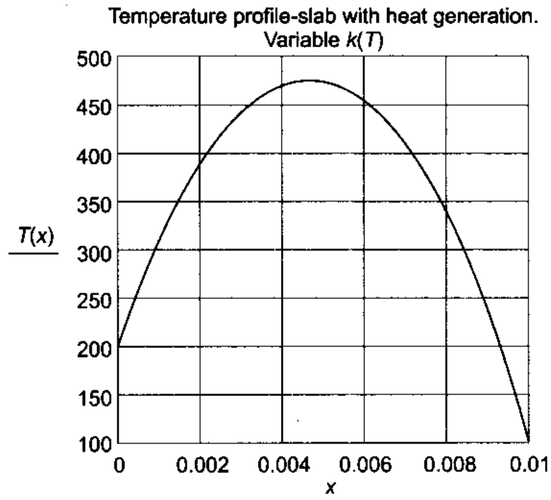

FIGURE Example 5.3(b)

Solution. See Fig. Example 5.4.

Data:

$L := 0.01$ m $\quad A := 1$ m^2 $\quad k := 25$ W/(mK)

$q_g := 0.3 \times 10^6$ W/m^3 $\quad T_a := 92$°C $\quad h_a := 500$ W/(m^{2}K)

By observation, we know that maximum temperature occurs on the insulated wall; this is so because, the heat generated in the wall is constrained to flow from left to right since the left face is insulated and for this to occur, temperature on the left must be higher than that on the right.

We can directly apply Eq. 5.14 and put $x = 0$ in that equation to get T_{max}.

We have, from eqn. 5.14

$$T(x) := T_a + \frac{q_g \cdot L}{h_a} + \frac{q_g}{2 \cdot k} \cdot (L^2 - x^2) \qquad ...(5.14)$$

Maximum temperature in the wall: (occurs at the insulated surface i.e. at $x = 0$)

Put $x = 0$ in Eq. (5.14):

$T(0) = 212$°C *(maximum temperature in the wall, occurs on the insulated left surface.)*

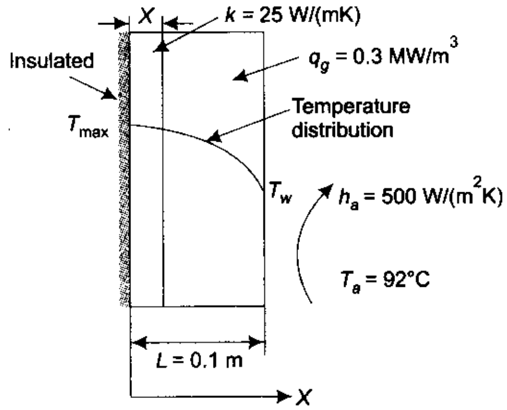

FIGURE Example 5.4 Plane slabe with internal heat generation, one side is insulated

Temperature at the surface exposed to the fluid:

Put $x = 0.1$ in Eq. (5.14):

$T(0.1) = 152$°C *(temperature at the surface exposed to the fluid.)*

To draw the temperature profile:

Using Mathcad, this is very easy. First, define a range variable x, varying from 0 to 0.1 m, with an increment of 0.005 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. Fig. Ex. 5.4(b).

$x := 0,. 0.005, ..., 0.1$ *(define a range variable x...starting value = 0, next value = 0.005 m and last value = 0.1 m)*

Note: It may be observed from the graph that the maximum temperature is 212°C and it occurs at $x = 0$ and at $x = 0.1$ m the temperature is 152°C.

Example 5.5. The exposed surface ($x = 0$) of a plane wall of thermal conductivity k, is subjected to microwave radiation that causes volumetric heating to vary as: $q_g(x) = q_o(1 - x/L)$ where q_o(W/m^3) is a constant. The boundary at $x = L$ is

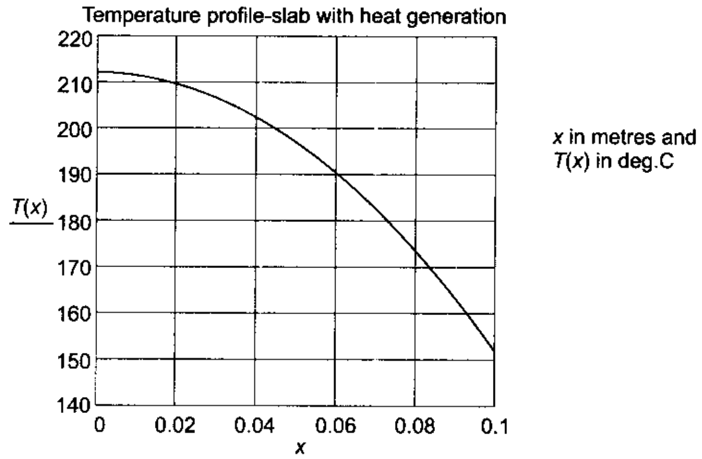

FIGURE Example 5.4(b)

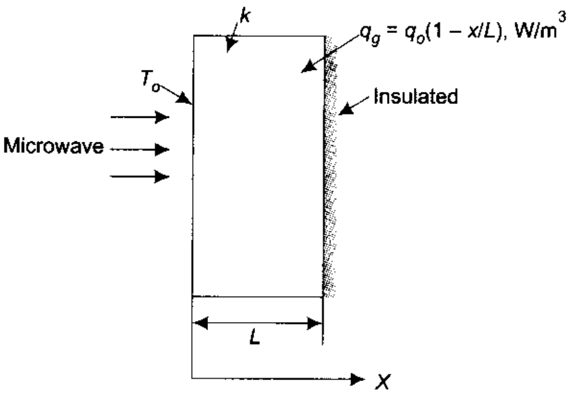

FIGURE Example 5.5 Plane slab with variable heat generation rate, one side insulated

perfectly insulated, while the exposed surface is maintained at a constant temperature, T_o. Determine the temperature distribution $T(x)$ in terms of x, L, k, q_o and T_o.

Solution. See Fig. Example 5.5.

Here, the heat generation rate is not uniform throughout the volume, but varies with position.

For the assumption of one-dimensional, steady state conduction with constant k, and the internal heat generation at the specified rate, the governing differential equation is:

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0 \quad \text{...(a)}$$

Substitute for q_g: $$\frac{d^2T}{dx^2} + \frac{q_o}{k}\cdot\left(1 - \frac{x}{L}\right) = 0$$

Integrating, $$\frac{dT}{dx} + \frac{q_o \cdot x}{k} - \frac{q_o \cdot x^2}{2 \cdot k \cdot L} = C_1 \quad \text{...(b)}$$

Again, integrating, $$T(x) + \frac{q_o \cdot x^2}{2 \cdot k} - \frac{q_o \cdot x^3}{6 \cdot k \cdot L} = C_1 x + C_2 \quad \text{...(c)}$$

where, C_1 and C_2 are constants of integration.

Eq. c gives the temperature distribution. C_1 and C_2 are obtained by applying the boundary conditions:

B.C. (i): at $x = 0$, $T = T_o$

B.C. (ii): at $x = L$, $dT/dx = 0$, since right face is insulated

B.C. (i) and Eq. c gives: $C_2 = T_o$

B.C. (ii) and Eq. b gives: $C_1 = \frac{q_o \cdot L}{2 \cdot k}$

Substituting C_1 and C_2 back in Eq. c:

$$T(x) = \frac{q_o \cdot x^3}{6 \cdot k \cdot L} - \frac{q_o \cdot x^2}{2 \cdot k} + \frac{q_o \cdot L \cdot x}{2 \cdot k} + T_o$$

i.e. $$T(x) = T_o + \frac{q_o \cdot L \cdot x}{2 \cdot k}\cdot\left(1 - \frac{x}{L} + \frac{1}{3}\cdot\frac{x^2}{L^2}\right) \quad \text{...(d)}$$

Eq. d is the desired relation for temperature distribution as a function of x, L, k, q_o and T_o.

Example 5.6. A copper conductor (k = 380 W/(mC), $\rho = 2 \times 10^{-8}$ ohm × m), 8 mm diameter and 1 m long, connects two large plates. One face is maintained at 30°C and the other face, at 50°C. Space between the plates is filled with an insulation.

(i) What is the maximum temperature and its location if the maximum current flowing is 150 A?

(ii) Calculate the heat dissipation to LHS and RHS

(iii) Draw the temperature profile.

Solution. See Figure Example 5.6.

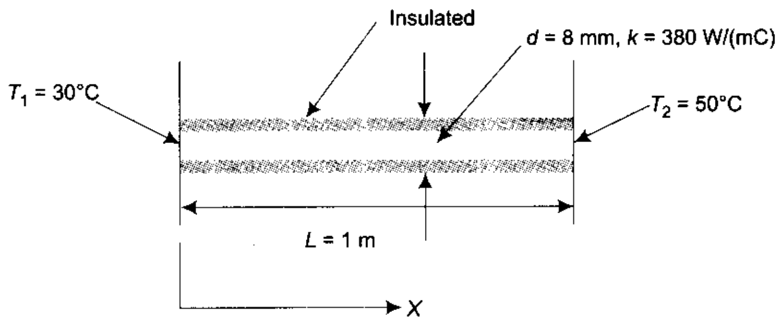

FIGURE Example 5.6 Rod connected between two plates

Data:

$L := 1.0$ m $\quad d := 0.008$ m $\quad T_1 := 30$°C $\quad T_2 := 50$°C $\quad k := 380$ W/(mC) $\quad \rho := 2 \times 10^{-8}$ Ohm × m

$I := 150$ Amp $\quad A := \dfrac{\pi \cdot d^2}{4}$ m^2 i.e. $A := 5.027 \times 10^{-5}$ m^2

Obviously, maximum temperature will occur at a location nearer to the end at 50°C.

Since the bar is laterally insulated, it is a case of one-dimensional conduction in the X-direction, in steady state, with heat generation and constant k.

So, the controlling differential equation is:

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

Integrating:

$$\frac{dT}{dx} + \frac{q_g \cdot x}{k} = C_1$$

Integrating again:

$$T(x) = \frac{-q_g \cdot x^2}{2 \cdot k} + C_1 \cdot x + C_2 \qquad \text{...(b)}$$

Eq. b gives the temperature distribution in the bar.

Apply the B.C.'s to get C_1 and C_2, the constants of integration.

B.C. (i): at $x = 0$, $T = 30$°C

B.C. (ii): at $x = 1$ m, $T = 50$°C

To calculate q_g:

$$q_g = \frac{Q}{\text{Volume}} = \frac{I^2 \cdot R}{\text{Volume}}, \text{ where Q is the heat generated.}$$

Resistance R:

$$R := \frac{\rho \cdot L}{A}$$

i.e.

$$R = 3.979 \times 10^{-4} \text{ Ohm} \qquad \textit{(resistance of the rod)}$$

Therefore,

$$q_g := \frac{I^2 \cdot R}{A \cdot L} \text{ W/m}^3 \qquad \textit{(heat generation rate due to Joule heating)}$$

i.e. $q_g = 1.78104 \times 10^5$ W/m^3 *(heat generated rate)*

B.C.(i) and Eq. b gives: $C_2 = 30$

B.C.(ii) and Eq. b gives: $C_1 := \dfrac{50 + \dfrac{q_g \cdot L^2}{2 \cdot k} - C_2}{L}$

Substituting and simplifying $C_1 = 254.347$

Substituting C_1 and C_2 back in Eq. b

$$T(x) = \frac{-(1.78104 \times 10^5) \cdot x^2}{2 \times 380} + 254.347 \cdot x + 30$$

i.e. $T(x) = -234.347 \cdot x^2 + 254.347 \cdot x + 30$ ...(c)

Eq. c is the desired eqn. c for temperature distribution.

Location and value of maximum temperature:

Location of maximum temperature is obtained by differentiating Eq. c w.r.t. x and equating to zero:

i.e. $-234.347 \cdot (2x) + 254.347 = 0$

i.e. $x := \dfrac{254.347}{234.347 \times 2}$

i.e. $x = 0.543$ m ***(location of maximum temperature...(this is the distance from LHS))***

Value of maximum temperature:

Substitute this value of x in Eq c

$T(0.543) = 99.013$°C ***(value of maximum temperature)***

Heat dissipated to LHS and RHS:

Since the temperature profile is known, get $T'(x) = dT(x)/dx$ at $x = 0$ and $x = L$, and then apply Fourier's law at $x = 0$ and $x = L$, to get Q_{left} and Q_{right}:

$T'(x) := \dfrac{d}{dx} T(x)$ *(define $T'(x)$, the first derivative of $T(x)$ w.r.t. x)*

Therefore, $T'(0) = 254.347$ C/m *(dT/dx at $x = 0$, ...i.e. at LHS)*

and, $T'(1) = -214.347$ C/m *(dT/dx at $x = 1$ m, ...i.e. at RHS)*

So, we have:

$Q_{left} := -k \cdot A \cdot T'(0)$ W *(define Q_{left})*

i.e. $Q_{left} = -4.858$ w ***(heat dissipated from left end.)***

Note: Negative sign of Q indicates that heat is flowing from right to left, i.e. in negative X-direction.

And, $Q_{right} := -k \cdot A \cdot T'(1)$, W *(define Q_{right})*

i.e. $Q_{right} = 4.094$ W ***(heat dissipated from right end.)***

Check: Sum of the heat dissipated from left and right ends must be equal to the total heat generated in the bar:

$Q_{tot} := |Q_{left}| + |Q_{right}|$ W *(define Q_{tot})*

i.e. $Q_{tot} = 8.952$ W *(total heat dissipated)*

Now, $Q_{gen} := I^2 R$ W *(heat generated by Joule heating)*

i.e. $Q_{gen} = 8.952$ W ***(checks with Q_{tot}.)***

To draw the temperature profile:

First, define a range variable x, varying from 0 to 1 m, with an increment of 0.01 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.6(b).

$x := 0, 0.01, \ldots, 1$ *(define a range variable x...starting value = 0, next value = 0.01 m and last value = 1 m)*

Note from the graph that maximum temperature of 99.01°C is reached at $x = 0.543$ m, i.e. beyond the mid-point, towards the right end.

5.3 Cylinder with Uniform Internal Heat Generation

There are several applications of cylindrical geometry with internal heat generation, e.g. current carrying conductors, nuclear fuel rods, chemical reactors, etc. We shall consider solid cylinders as well as hollow cylinders with different types of boundary conditions.

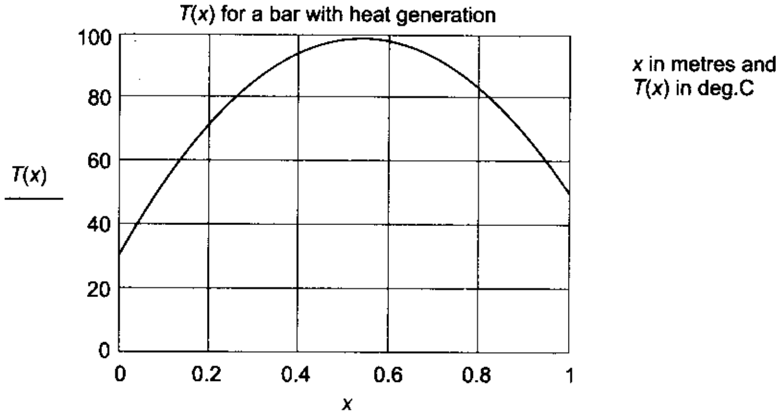

FIGURE Example 5.6(b)

5.3.1 *Solid Cylinder with Internal Heat Generation*

Consider a solid cylinder of radius, R and length, L. There is uniform heat generation within its volume at a rate of q_g (W/m^3). Let the thermal conductivity, k be constant.

See Fig. 5.5.

We would like to analyse this system for temperature distribution and maximum temperature attained.

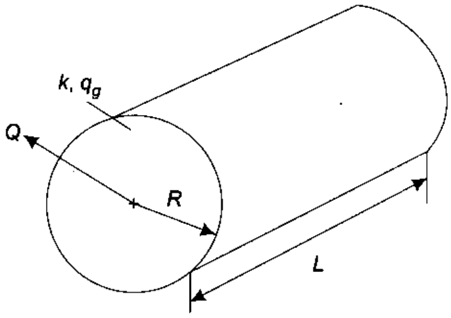

FIGURE 5.5(a) Cylindrical system with heat generation

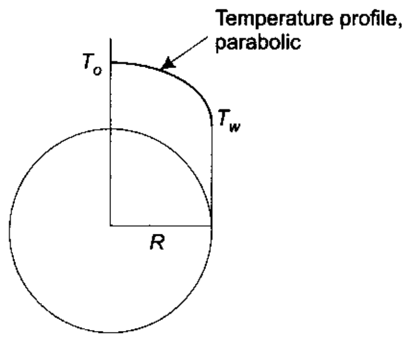

FIGURE 5.5(b) Variation of temperature along the radius

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the r direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Uniform internal heat generation rate, q_g (W/m^3).

With the above stipulations, the general differential equation in cylindrical coordinates (see Eq. 3.17) reduces to:

$$\frac{d^2T}{dr^2} + \frac{1}{r} \cdot \frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

Multiplying by r: $r \cdot \frac{d^2T}{dr^2} + \frac{dT}{dr} + \frac{q_g \cdot r}{k} = 0$

i.e. $$\frac{d}{dr}\left(r\cdot\frac{dT}{dr}\right) = \frac{-q_g\cdot r}{k}$$

Integrating: $$r\cdot\frac{dT}{dr} = \frac{-q_g\cdot r^2}{2\cdot k} + C_1$$

i.e. $$\frac{dT}{dr} = \frac{-q_g\cdot r}{2\cdot k} + \frac{C_1}{r} \quad ...(b)$$

Integrating again: $$T(r) = \frac{-q_g\cdot r^2}{4\cdot k} + C_1\cdot \ln(r) + C_2 \quad ...(5.18)$$

Eq. 5.18 is the general relation for temperature distribution along the radius, for a cylindrical system, with uniform heat generation.

C_1 and C_2, the constants of integration are obtained by applying the boundary conditions.

(Remember Eq. 5.18, since the same equation will be the starting point in the analysis of hollow cylinders too, with different boundary conditions.)

In the present case, B.C.'s are:

B.C. (i): at $r = 0$, $dT/dr = 0$, i.e. at the centre of the cylinder, temperature is finite and maximum (i.e. $T_o = T_{max}$) because of symmetry (heat flows from inside to outside radially).

B.C. (ii): at $r = R$, i.e. at the surface, $T = T_w$

From B.C. (i) and Eq. b, we get: $C_1 = 0$

From B.C. (ii) and Eq. 5.18, we get:

$$T_w = \frac{-q_g\cdot R^2}{4\cdot k} + C_2$$

i.e. $$C_2 = T_w + \frac{q_g\cdot R^2}{4\cdot k}$$

Substituting C_1 and C_2 in Eq. 5.18

$$T(r) = \frac{-q_g\cdot r^2}{4\cdot k} + T_w + \frac{q_g\cdot R^2}{4\cdot k}$$

i.e. $$T(r) = T_w + \frac{q_g}{4\cdot k}\cdot(R^2 - r^2) \quad ...(5.19)$$

Eq. 5.19 is the relation for temperature distribution in terms of the surface temperature, T_w. Note that this is a parabolic temperature profile, as shown in Fig. 5.11(b).

Maximum temperature:

Maximum temperature occurs at the centre, because of symmetry considerations (i.e. heat flows from the centre radially outward in all directions; therefore, temperature at the centre must be a maximum.)

Therefore, putting $r = 0$ in Eq. 5.19:

$$T_{max} = T_w + \frac{q_g\cdot R^2}{4\cdot k} \quad ...(5.20)$$

From Eq. 5.19 and 5.20,

$$\frac{T - T_w}{T_{max} - T_w} = 1 - \left(\frac{r}{R}\right)^2 \quad ...(5.21)$$

Eq. 5.21 is the non-dimensional temperature distribution for the solid cylinder with heat generation.

Convection boundary condition:

In many practical applications, heat is carried away at the boundaries by a fluid at a temperature T_a flowing on the surface with a convective heat transfer coefficient, h (e.g. current carrying wire cooled by ambient air). Then, mostly, it is the fluid temperature that is known and not the surface temperature, T_w, of the cylinder. In such cases, we relate the wall temperature and fluid temperature by an energy balance at the surface, i.e. heat generated and conducted from within the body to the surface is equal to the heat convected away by the fluid at the surface.

i.e. $$\pi \cdot R^2 \cdot L \cdot q_g = h \cdot (2 \cdot \pi \cdot R \cdot L) \cdot (T_w - T_a)$$

i.e. $$T_w = T_a + \frac{q_g \cdot R}{2 \cdot h} \qquad \text{...(c)}$$

Substituting c in Eq. 5.19:

$$T(r) = T_a + \frac{q_g \cdot R}{2 \cdot h} + \frac{q_g}{4 \cdot k} \cdot (R^2 - r^2) \qquad \text{...(5.22)}$$

Again, for maximum temperature put $r = 0$ in Eq. 5.22:

$$T_{max} = T_a + \frac{q_g \cdot R}{2 \cdot h} + \frac{q_g \cdot R^2}{4 \cdot k} \qquad \text{...(5.23)}$$

Eq. 5.23 gives maximum temperature in the solid cylinder in terms of the fluid temperature, T_a.

5.3.1.1 Alternative analysis. In the alternative method, which is simpler, instead of starting with the general differential equation, we derive the above equations from physical considerations. See Fig. 5.6.

Let us write an energy balance with an understanding that at any radius r, the amount of heat generated in the volume within $r = 0$ and $r = r$, must move outward by conduction.

i.e. at any radius r, we write the energy balance:

$$q_g \cdot \pi \cdot r^2 \cdot L = -k \cdot (2 \cdot \pi \cdot r \cdot L) \cdot \frac{dT}{dr} \qquad \text{...(a)}$$

$$dT = \frac{-q_g}{2 \cdot k} \cdot r \cdot dr$$

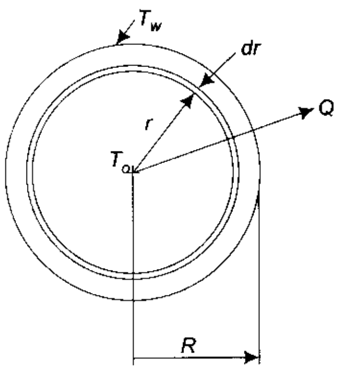

FIGURE 5.6 Solid cylinder with heat generation

Integrating:

$$\int dT = \frac{-q_g}{2 \cdot k} \int r dr$$

i.e. $$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C \qquad \text{...(b)}$$

Eq. b gives the temperature distribution along the radius.

Get the constant of integration, C from the B.C.: at $r = R$, $T = T_w$

i.e. $$C = T_w + \frac{q_g \cdot R^2}{4 \cdot k}$$

Substituting C back in Eq. b:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + T_w + \frac{q_g \cdot R^2}{4 \cdot k}$$

i.e.
$$T(r) = T_w + \frac{q_g}{4 \cdot k} \cdot (R^2 - r^2) \qquad \text{...(c)}$$

Eq. c gives the temperature distribution along the radius, in terms of the surface temperature of the cylinder. Note that Eq. c is the same as Eq. 5.19 derived earlier.

In many applications, temperature drop between the centre line (where maximum temperature occurs) and the surface is important (e.g. in nuclear fuel rods, to ensure that the fuel rod does not melt). Then, from Eq. c, putting $r = 0$:

$$T_o - T_w = \frac{q_g \cdot R^2}{4 \cdot k} \qquad \text{...(5.24)}$$

Eq. 5.24 is important; it gives the maximum temperature difference in a solid cylinder with heat generation. Knowing T_w, one can easily find out T_o (= T_{max}).

Compare Eq. 5.24 with Eq. 5.9, derived earlier for the maximum temperature difference in a slab with uniform heat generation.

5.3.1.2 Analysis with variable thermal conductivity. In the above analysis, thermal conductivity of the material was assumed to be constant. Now, let us make an analysis when the thermal conductivity varies linearly with temperature as:

$$k(T) = k_o(1 + \beta T),$$

where, k_o and β are constants.

Again, considering Fig. 5.6, we have from heat balance (see Eq. a above):

$$q_g \cdot \pi \cdot r^2 \cdot L = -k(T) \cdot (2 \cdot \pi \cdot r \cdot L) \cdot \frac{dT}{dr} \qquad \text{...(a)}$$

i.e.
$$k(T) \cdot dT = \frac{-q_g}{2} \cdot r \cdot dr$$

Substituting for $k(T)$ and integrating:

$$\int k_o \cdot (1 + \beta \cdot T) dT = \frac{-q_g}{2} \cdot \int r dr$$

i.e.
$$T + \frac{\beta \cdot T^2}{2} = \frac{-q_g \cdot r^2}{4 \cdot k_o} + C \qquad \text{...(e)}$$

C is determined from the B.C.: at $r = 0$, $T = T_o$

We get:

$$C = T_o + \frac{\beta \cdot T_o^2}{2}$$

Substituting C in Eq. e:

$$\frac{\beta \cdot T^2}{2} + T + \frac{q_g \cdot r^2}{4 \cdot k_o} - T_o - \frac{\beta \cdot T_o^2}{2} = 0 \qquad \text{...(f)}$$

Eq. f is a quadratic in T. Its positive root is given by:

$$T(r) = \frac{-1 + \sqrt{1 - 4 \cdot \frac{\beta}{2} \cdot \left(\frac{q_g \cdot r^2}{4 \cdot k_o} - T_o - \frac{\beta \cdot T_o^2}{2} \right)}}{2 \cdot \frac{\beta}{2}}$$

i.e. $$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta^2} + T_o^2 + \frac{2 \cdot T_o}{\beta}\right) - \frac{q_g \cdot r^2}{2 \cdot \beta \cdot k_o}}$$

i.e. $$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_o\right)^2 - \frac{q_g \cdot r^2}{2 \cdot \beta \cdot k_o}} \quad ...(5.25)$$

Eq. 5.25 gives temperature distribution in a solid cylinder with internal heat generation and linearly varying k. Compare this equation with that obtained for a slab, with temperature at either side being the same, i.e. Eq. 5.10.

Eq. 5.25 gives $T(r)$ in terms of T_o (i.e. T_{max} at $r = 0$).

If we need $T(r)$ in terms of T_w: in Eq. e, C is determined from:

B.C.: at $r = R$, $T = T_w$

we get:

$$C = T_w + \frac{\beta \cdot T_w^2}{2} + \frac{q_g \cdot R^2}{4 \cdot k_o}$$

Substituting C in Eq. e, we get a quadratic in T, and solving we get, for temperature distribution:

$$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_w\right)^2 + q_g \cdot \frac{(R^2 - r^2)}{2 \cdot \beta \cdot k_o}} \quad ...(5.26)$$

5.3.1.3 Current carrying conductor. This is a very important practical application. Cooling of current carrying conductors enhances their current carrying capacity. Knowledge of temperature distribution is required to make sure that temperatures leading to 'burn out' of the conductor are not reached. Conductors have to operate safely in superconducting magnets, transformers, motors and electrical machinery, since sudden failure of conductor may lead to conditions that are unsafe to the operator as well as the machine.

In the case of current carrying conductors, uniform internal heat generation occurs due to Joule heating.

Consider a conductor of cross-sectional area, A_c and length, L. Let the current carried be I (A). Let the electrical resistivity of the material be ρ (Ohm × m).

Then, heat generated per unit volume = Q_g/Volume of conductor,

where, Q_g is the total heat generated (W).

$$Q_g = I^2 \cdot R \text{ where } R = \text{electrical resistance of wire, (Ohms)}$$

But, $$R = \frac{\rho \cdot L}{A_c}$$

Therefore, $$q_g = \frac{I^2 \cdot R}{A_c \cdot L} = \frac{I^2 \cdot \left(\frac{\rho \cdot L}{A_c}\right)}{A_c \cdot L} = \left(\frac{I}{A_c}\right)^2 \cdot \rho, \text{ W/m}^3$$

$i = I/A_c$, is known as the 'current density'. Note its units: A/m^2

i.e. $$q_g = i^2 \cdot \rho = \frac{i^2}{k_e} \text{ where } k_e = \frac{1}{\rho} = \text{electrical conductivity, (Ohm m)}^{-1}$$

Therefore, temperature distribution in a current carrying wire (of solid, cylindrical shape) is given by Eq. 5.19, viz.

$$T(r) = T_w + \frac{q_g}{4 \cdot k} \cdot (R^2 - r^2) \quad ...(5.19)$$

Substituting for q_g, we get:

$$T(r) = T_w + \frac{i^2 \cdot \rho}{4 \cdot k} \cdot (R^2 - r^2) \quad \text{...(5.19a)}$$

Eq. 5.19a gives the temperature distribution in the current carrying wire, in terms of the surface temperature, T_w. Maximum temperature, which occurs at the centre, is obtained by putting $r = 0$ in Eq. 5.19 a. i.e.

$$T_{max} = T_w + \frac{i^2 \cdot \rho \cdot R^2}{4 \cdot k} \quad \text{....(5.20a)}$$

And, from Eqs. 5.19a and 5.20a, we get:

$$\frac{T - T_w}{T_{max} - T_w} = 1 - \left(\frac{r}{R}\right)^2$$

Note that the above equation for non-dimensional temperature distribution in a current carrying wire is the same as Eq. 5.21.

Example 5.7. (a) A 3.2 mm diameter stainless steel wire, 30 cm long has a voltage of 10 V impressed on it. The outer surface temperature of the wire is maintained at 93°C. Calculate the centre temperature of the wire. Take the resistivity of the wire as 70 micro-ohm × cm and the thermal conductivity as 22.5 W/(mK).

(b) The heated wire in the above example is submerged in a fluid maintained at 93°C. The convection heat transfer coefficient is 5.7 kW/(m²K). Calculate the centre temperature of the wire.

Solution. See Figure Example 5.7.

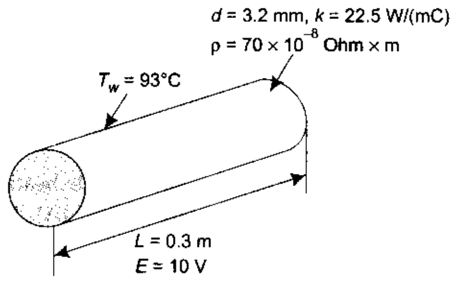

FIGURE Example 5.7(a) Wire with an impressed voltage, T_w is known

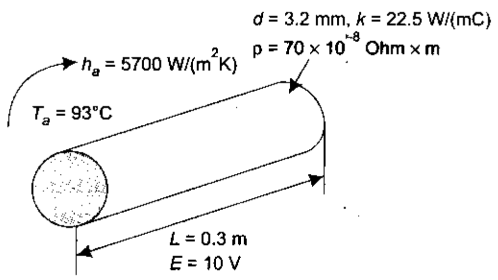

FIGURE Example 5.7(b) Wire with an impressed voltage, T_a is known

Data:

$d_o := 0.0032$ m $\quad R := \frac{d_o}{2}$ m i.e. $R = 1.6 \times 10^{-3}$ m $\quad L := 0.3$ m $\quad \rho := 70 \times 10^{-8}$ Ohm × m $\quad k := 22.5$ W/(mC)

$T_w := 93°C \quad T_a := 93°C \quad h := 5700$ W/(m²C) $\quad E := 10$ V $\quad A := \pi \cdot \frac{d_o^2}{4}$, m² $\quad$ Resistance $:= \rho \cdot \frac{L}{A}$ Ohm

i.e. $\quad$ Resistance = 0.026 Ohm $\quad$ *(electrical resistance of the wire)*

$$P := \frac{E^2}{\text{Resistance}} \text{ W} \quad \textit{(define power generated due to current flow)}$$

i.e. $\quad P = 3.83 \times 10^3$ W $\quad$ *...power generated*

$$q_g := \frac{P}{A \cdot L} \text{ W/m}^3 \quad \textit{(define the internal heat generation rate)}$$

i.e. $\quad q_g := 1.587 \times 10^9$ W/m³ $\quad$ *(the internal heat generation rate)*

Case (a): Wire surface temperature is given;

To calculate centre temperature (i.e. maximum temperature):

We have, from Eq. 5.20:

$$T_{max} := T_w + q_g \cdot R^2 \cdot \frac{1}{4 \cdot k}$$

i.e. $T_{max} = 138.15°C$ *...centre temperature of wire*

Case (b): Wire submerged in a fluid;

To calculate the centre temperature (i.e. maximum temperature):

We have, from Eq. 5.23:

$$T_{max} := T_a + \frac{q_g \cdot R}{2 \cdot h} + \frac{q_g}{4 \cdot k} \cdot R^2$$

i.e. $T_{max} = 360.929°C$ *...centre temperature of wire*

Example 5.8. Meat rolls of 25 mm diameter, $k = 1$ W/(mC) are heated by microwave heating. Centre temperature of the roll is 90°C. Surrounding temperature is at 30°C. Heat transfer coefficient at the surface is 25 W/(m^{2}C). Find the microwave heating capacity required in W/m^3.

Solution.

Data:

$R := 0.0125$ m $\quad k := 1$ W/(mC) $\quad T_o := 90°C \quad T_a := 30°C \quad h := 25$ W/(m^{2}C)

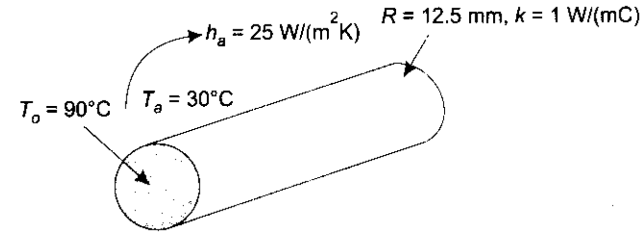

FIGURE Example 5.8 Microwave heating of meat roll

Remember that for the cylindrical roll, maximum temperature occurs at the centre.

We have, from Eq. 5.23:

$$T_o = T_a + \frac{q_g \cdot R}{2 \cdot h} + \frac{q_g \cdot R^2}{4 \cdot k}$$

i.e.

$$T_o = T_a + q_g \cdot \left(\frac{R}{2 \cdot h} + \frac{R^2}{4 \cdot k} \right)$$

i.e.

$$q_g := \frac{T_o - T_a}{\left(\frac{R}{2 \cdot h} + \frac{R^2}{4 \cdot k} \right)} \text{ W/m}^3 \qquad \textit{(define } q_g\textit{)}$$

i.e. $q_g = 2.07 \times 10^5$ W/m^3 (= 207.6 kW/m^3 ...***required microwave heating capacity.***)

Example 5.9. A long cylindrical rod of diameter 200 mm with $k = 0.5$ W/(mK) experiences uniform volumetric heat generation of 24,000 W/m^3. The rod is encapsulated by a circular sleeve having an outer diameter of 400 mm and k of 4 W/(mK). Outer surface of the sleeve is exposed to cross flow of air at 27°C with convection coefficient of 25 W/(m^{2}K).

(i) Find the temperature at the interface between the rod and the sleeve and on the outer surface.
(ii) What is the temperature at the centre of the rod?
(iii) What is the temperature at mid-radius of the rod?
(iii) Sketch the temperature distribution.

Solution. See Figure Example 5.9.

Data:

$R_1 := 0.1$ m $\quad R_2 := 0.2$ m $\quad L := 1$ m $\quad k_1 := 0.5$ W/(mK) $\quad k_2 := 4$ W/(mK) $\quad T_a := 27°C$

$h_a := 25$ W/(m^{2}K) $\quad q_g := 24000$ W/m^3

Let T_0, T_1, and T_2 be the centre temperature of the rod, interface temperature between the rod and the sleeve, and the outer surface temperature of sleeve, respectively, as shown in Fig. 5.16.

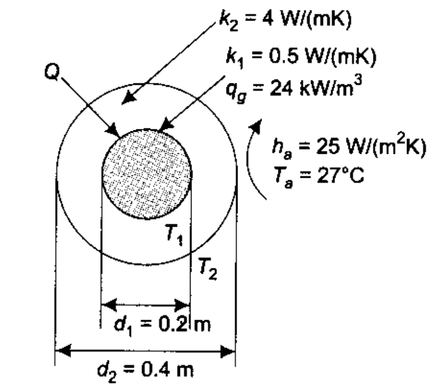

FIGURE Example 5.9 Encapsulated rod with heat generation

For the inner cylindrical rod with heat generation, we have, from Eq. 5.20:

$$T_o = T_1 + \frac{q_g \cdot R_1^2}{4 \cdot k_1} \quad \text{...(a)}$$

Here, however, T_1 is not presently known. But T_1 is related to the known ambient temperature T_a by considering the steady state heat transfer from the inner cylinder through the outer sleeve by conduction and then to the ambient fluid by convection.

i.e. $$Q = \frac{T_1 - T_a}{R_{sleeve} + R_{conv}} \quad \text{...(b)}$$

where, R_{sleeve} = thermal resistance of sleeve, and R_{conv} = convective resistance on outer surface of sleeve

First, find Q, the steady state heat transfer rate = heat generation rate in the cylinder

i.e. $Q := \pi \cdot R_1^2 \cdot L \cdot q_g$ W *(define Q, the total heat generated)*

i.e. $Q = 753.982$ W *(total heat generated rate.)*

Thermal resistances:

$$R_{sleeve} := \frac{\ln\left(\frac{R_2}{R_1}\right)}{2 \cdot \pi \cdot k_2 \cdot L} \text{ C/W}$$ *(define thermal resistance of sleeve)*

i.e. $R_{sleeve} = 0.028$ C/W *(thermal resistance of sleeve)*

and, $$R_{conv} = \frac{1}{h_a \cdot (2 \cdot \pi \cdot R_2 \cdot L)} \text{ C/W}$$ *(convective resistance on the outer surface of sleeve)*

i.e. $R_{conv} = 0.032$ C/W *(convective resistance on the outer surface of sleeve.)*

Temperatures T_1, T_2 and T_o:

From Eq. b:

$$T_1 := Q \cdot (R_{sleeve} + R_{conv}) + T_a \text{ °C}$$ *(define T_1 the interface temperature between cylinder and sleeve)*

i.e. $T_1 = 71.794$°C ***(the interface temperature between cylinder and sleeve.)***

To find T_2:

We have: $$Q = \frac{T_1 - T_2}{R_{sleeve}} \text{ W}$$ *...applying Ohm's law to the sleeve*

i.e. $T_2 := T_1 - Q \cdot R_{sleeve}$, °C *...define T_2*

i.e. $T_2 = 51$°C ***(temperature on the outer surface of sleeve.)***

To find T_o:

From Eq. a:

$$T_o := T_1 + \frac{q_g \cdot R_1^2}{4 \cdot k_1} \text{ °C}$$ *(define T_o, the centre temperature)*

i.e. $T_o = 191.794$°C ***(centre temperature of cylinder.)***

Temperature at the mid-radius of the rod, i.e. at r = 0.05 m:

For a cylinder with heat generation, temperature distribution is given by Eq. 5.19:

i.e. $$T(r) = T_w + \frac{q_g}{4 \cdot k} \cdot (R^2 - r^2) \quad \text{...(5.19)}$$

For the present case, this equation becomes:

$$T(r) := T_1 + \frac{q_g}{4 \cdot k_1} \cdot (R_1^2 - r^2)$$ *(define T(r))*

Therefore, $T(0.05) = 161.794$°C ***(temperature at mid-radius of the rod.)***

To sketch the temperature profile:

Temperature profile for the rod with heat generation is given by:

$$T(r) = T_1 + \frac{q_g}{4 \cdot k_1} \cdot (R_1^2 - r^2)$$

And, temperature profile for the cylindrical shell of sleeve (with no heat generation) is given by Eq. 4.34, i.e.

$$t(r_s) = T_1 + \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)} \cdot \ln\left(\frac{r_s}{R_1}\right)$$ where r_s = any radius within the sleeve.

To sketch the temperature profile in the rod, define a range variable r, varying from 0 to 0.1 m, with an increment of 0.005 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.9(b).

$r := 0, 0.005, \ldots, 0.1$ *(define a range variable r..starting value = 0, next value = 0.005 m and last value = 0.1 m)*

To sketch the temperature profile in the sleeve, define a range variable r_s, varying from 0.1 to 0.2 m, with an increment of 0.005 m. Then, in the above graph, on the x-axis place holder, put a comma after r, and enter r_s and on the y-axis place holder, put a comma after $T(r)$ and enter $t(r_s)$. Click anywhere outside the graph region, and immediately both the graphs appear.

$r_s := 0.1, 0.105, \ldots, 0.2$ *(define a range variable r_s..starting value = 0.1, next value = 0.105 m and last value = 0.2 m)*

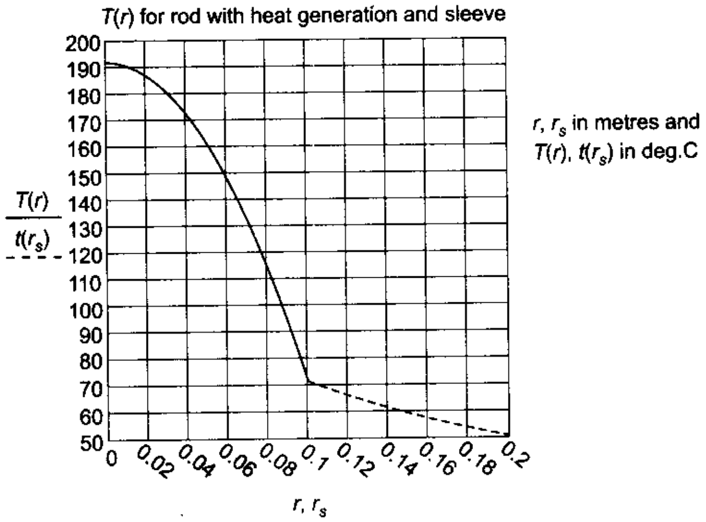

FIGURE Example 5.9(b)

In the above figure from $r = 0$ to $r = 0.1$ m, the graph shows the temperature profile within the solid rod with internal heat generation; from the radius of 0.1 m to 0.2 m, the graph shows the temperature profile within the cylindrical sleeve placed over the rod.

Note that at $r = 0$, $T_o = 191.8$°C, at $r = 0.1$ m, $T_1 = 71.8$°C and at $r = 0.2$ m, $T_2 = 51$°C.

5.3.2 *Hollow Cylinder with Heat Generation*

Hollow cylinder geometry has significant practical applications. Many times, nuclear fuel rods are made of hollow cylinder geometry where the heat generated is carried away by a (liquid metal) coolant flowing either on the inside or outside the tubes. Hollow electrical conductors of cylindrical shape are used for high current carrying applications, where again, cooling is done by a fluid flowing on the inside. There are annular reactors, insulated either from inside or outside, used in chemical processes.

We shall study heat transfer in a hollow cylindrical system, with different boundary conditions.

5.3.2.1 Hollow cylinder with the inside surface insulated. Consider steady state, one-dimensional heat transfer in a hollow cylinder of length L, inside radius r_i and outside radius r_o, with a uniform internal heat generation rate of q_g (W/m^3). Thermal conductivity, k is constant. Let the inside surface be perfectly insulated; that means, all the heat generated in the cylindrical shell has to move only outwards, in the positive r-direction. Let the temperatures on the inside and outside surfaces be T_i and T_o respectively. See Fig. 5.7.

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the r direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Uniform internal heat generation rate, q_g (W/m^3).

With the above stipulations, the general differential equation in cylindrical coordinates (see Eq. 3.17) reduces to:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\cdot\frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad ...(a)$$

Integrating Eq. a twice, we get the general solution for temperature distribution, namely, Eq. 5.18, as done in section 5.3.1:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \qquad ...(5.18)$$

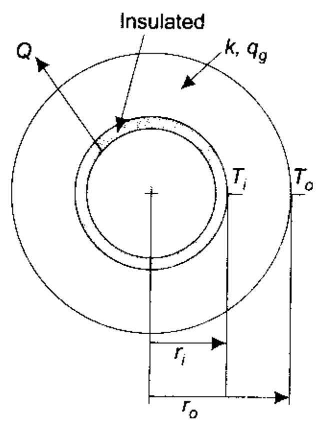

FIGURE 5.7 Hollow cylinder with heat generation, inside surface insulate

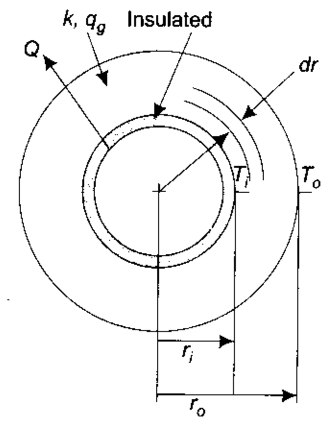

FIGURE 5.8 Hollow cylinder with heat generation, inside surface insulated

Eq. 5.18 is the general relation for temperature distribution along the radius, for a cylindrical system, with uniform heat generation.

C_1 and C_2, the constants of integration are obtained by applying the boundary conditions.

In the present case the B.C.'s are:

B.C.(i): at $r = r_i$ $T = T_i$ and $dT/dx = 0$ (since inner surface is insulated), and

B.C.(ii): at $r = r_o$ $T = T_o$

Get C_1 and C_2 from these B.C.'s and substitute back in Eq. 5.18 to get the temperature distribution. This is left as an exercise for the student (See Example 5.11 for procedure of working out a numerical problem).

We shall, however, derive the expression for temperature distribution by a simpler method of physical consideration and heat balance:

Alternative Method

See Fig. 5.8.

Consider any radius r in the cylindrical shell as shown.

Since the inside surface is insulated, heat generated within the volume between $r = r_i$ and $r = r$, must travel only outward; and, this heat must be equal to the heat conducted away from the surface at radius r.

Writing this heat balance,

$$q_g \cdot \pi \cdot (r^2 - r_i^2) \cdot L = -k \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{dT}{dr}$$

where, dT/dr is the temperature gradient at radius r.

i.e.
$$dT = \frac{q_g \cdot r_i^2}{2 \cdot k} \cdot \frac{dr}{r} - \frac{q_g}{2 \cdot k} r \cdot dr$$

Integrating
$$T(r) = \frac{q_g \cdot r_i^2}{2 \cdot k} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k} + C \quad \text{...(b)}$$

Eq. b is the general solution for temperature distribution.

The integration constant C is obtained by the B.C.:

At $r = r_o$, $T = T_o$

Applying this B.C. to Eq. b:

$$C = T_o + \frac{q_g \cdot r_o^2}{4 \cdot k} - \frac{q_g \cdot r_i^2}{2 \cdot k} \cdot \ln(r_o)$$

Substituting value of C back in Eq. b we get,

$$T(r) = \frac{q_g \cdot r_i^2}{2 \cdot k} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k} + T_o + \frac{q_g \cdot r_o^2}{4 \cdot k} - \frac{q_g \cdot r_i^2}{2 \cdot k} \cdot \ln(r_o)$$

i.e.
$$T(r) = T_o + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r} \right) - \left(\frac{r}{r_i} \right)^2 \right] \quad \text{...(5.27)}$$

Eq. 5.27 gives the temperature distribution in a hollow cylinder with heat generation, insulated on the inside surface, in terms of the outer wall temperature, T_o.

Putting $r = r_i$ and $T = T_i$ in Eq. 5.27, we get,

$$T_i = T_o + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i} \right) - 1 \right]$$

i.e.
$$T_i - T_o = \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i} \right) - 1 \right] \quad \text{...(5.28)}$$

Eq. 5.28 is important, since it gives the maximum temperature drop in the cylindrical shell, when there is internal heat generation and the inside surface is insulated.

If either of T_o or T_i is given in a problem, then the other temperature can be calculated using Eq. 5.28.

Convection boundary condition:

If heat is carried away at the outer surface by a fluid at a temperature T_a flowing on the surface with a convective heat transfer coefficient, h_a, then, it is the fluid temperature that is known and not the surface temperature, T_o. In such cases, we relate the surface temperature and fluid temperature by an energy balance at the surface, i.e. heat generated within the body and conducted to the outer surface is equal to the heat convected away by the fluid at the surface.

i.e.
$$q_g \cdot \pi \cdot \left(r_o^2 - r_i^2 \right) L = h_a \cdot 2 \cdot \pi \cdot r_o \cdot L \cdot (T_o - T_a)$$

i.e.
$$T_o = T_a + \frac{q_g \cdot \left(r_o^2 - r_i^2 \right)}{2 \cdot h_a \cdot r_o} \quad \text{...(c)}$$

Substituting the value of T_o from Eq. c in Eq. 5.27, we get:

$$T(r) = T_a + \frac{q_g \cdot (r_o^2 - r_i^2)}{2 \cdot h_a \cdot r_o} + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r} \right) - \left(\frac{r}{r_i} \right)^2 \right] \quad \text{...(5.29)}$$

Eq. 5.29 gives the temperature distribution in the cylindrical shell with heat generation, inside surface insulated, when the heat generated is carried away by a fluid flowing on the outer surface.

5.3.2.2 Analysis with variable thermal conductivity. In the above analysis, thermal conductivity of the material was assumed to be constant. Now, let us make an analysis when the thermal conductivity varies linearly with temperature as:

$$k(T) = k_o(1 + \beta T),$$

where, k_o and β are constants.

Again, considering Fig. 5.8, we have, from heat balance:

$$q_g \cdot \pi \cdot (r^2 - r_i^2) \cdot L = -k(T) \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{dT}{dr}$$

where, dT/dr is the temperature gradient at radius r.

i.e.

$$k(T) \cdot dT = \frac{q_g \cdot (r_i^2 - r^2)}{2 \cdot r} \cdot dr$$

Substituting for $k(T)$,

$$k_o \cdot (1 + \beta \cdot T) \cdot dT = \frac{q_g \cdot r_i^2}{2} \cdot \frac{dr}{r} - \frac{q_g}{2} r \cdot dr$$

Integrating,

$$T + \frac{\beta \cdot T^2}{2} = \frac{q_g \cdot r_i^2}{2 \cdot k_o} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k_o} + C \quad \text{...(d)}$$

In Eq. d, C is the integration constant. It is obtained by applying the B.C.,

At $r = r_i$, $T = T_i$

Applying this B.C. to Eq. d:

$$C = T_i + \frac{\beta \cdot T_i^2}{2} - \frac{q_g \cdot r_i^2}{2 \cdot k_o} \cdot \ln(r_i) + \frac{q_g \cdot r_i^2}{4 \cdot k_o}$$

Substituting value of C back in Eq. d:

$$\frac{\beta \cdot T^2}{2} + T - \left(\frac{q_g \cdot r_i^2}{2 \cdot k_o} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k_o} + T_i + \frac{\beta \cdot T_i^2}{2} - \frac{q_g \cdot r_i^2}{2 \cdot k_o} \cdot \ln(r_i) + \frac{q_g \cdot r_i^2}{4 \cdot k_o} \right) = 0$$

i.e.

$$\frac{\beta \cdot T^2}{2} + T - \left[\frac{-q_g \cdot r_i^2}{4 \cdot k_o} \cdot \left[\left(\frac{r}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r}{r_i} \right) - 1 \right] + T_i + \frac{\beta \cdot T_i^2}{2} \right] = 0 \quad \text{...(e)}$$

Eq. e is a quadratic in T. Its positive root is given by:

$$T(r) = \frac{-1 + \sqrt{1 + 4 \cdot \frac{\beta}{2} \cdot \left[\frac{-q_g \cdot r_i^2}{4 \cdot k_o} \cdot \left[\left(\frac{r}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r}{r_i} \right) - 1 \right] + T_i + \frac{\beta \cdot T_i^2}{2} \right]}}{2 \cdot \frac{\beta}{2}}$$

i.e. $$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta^2} + T_i^2 + \frac{2 \cdot T_i}{\beta}\right) - \frac{q_g \cdot r_i^2}{2 \cdot \beta \cdot k_o} \cdot \left[\left(\frac{r}{r_i}\right)^2 - 2 \cdot \ln\left(\frac{r}{r_i}\right) - 1\right]}$$

i.e. $$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_i\right)^2 - \frac{q_g \cdot r_i^2}{2 \cdot \beta \cdot k_o} \cdot \left[\left(\frac{r}{r_i}\right)^2 - 2 \cdot \ln\left(\frac{r}{r_i}\right) - 1\right]} \quad ...(5.30)$$

Eq. 5.30 gives the temperature distribution in a hollow cylinder with internal heat generation when the inside surface is insulated and the thermal conductivity varies linearly with temperature.

5.3.2.3 Hollow cylinder with the outside surface insulated. Consider steady state, one-dimensional heat transfer in a hollow cylinder of length L, inside radius r_i and outside radius r_o, with a uniform internal heat generation rate of q_g (W/m^3). Thermal conductivity, k is constant. Let the outside surface be perfectly insulated; that means, all the heat generated in the cylindrical shell has to move only inwards, in the negative r-direction. Let the temperatures on the inside and outside surfaces be T_i and T_o, respectively. See Fig. 5.9.

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the r direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Uniform internal heat generation rate, q_g (W/m^3).

With the above stipulations, the general differential equation in cylindrical coordinates (see Eq. 3.17) reduces to:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\cdot\frac{dT}{dr} + \frac{q_g}{k} = 0 \quad ...(a)$$

Integrating Eq. a twice, we get the general solution for temperature distribution, i.e. Eq. 5.18, as done in section 5.3.1:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \quad ...(5.18)$$

Eq. 5.18 is the general relation for temperature distribution along the radius, for a cylindrical system, with uniform heat generation.

C_1 and C_2, the constants of integration are obtained by applying the boundary conditions.

In the present case, the B.C.'s are:

B.C.(i): at $r = r_i$ $T = T_i$, and

B.C.(ii): at $r = r_o$ $T = T_o$, and $dT/dx = 0$ (since outer surface is insulated).

Get C_1 and C_2 from these B.C.'s and substitute back in Eq. 5.18 to get the temperature distribution. This is left as an exercise for the student (see Example 5.12. for procedure of working out a numerical problem).

We shall, however, derive the expression for temperature distribution by a simpler method of physical consideration and heat balance:

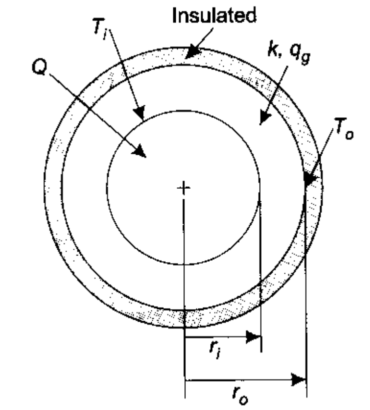

FIGURE 5.9 Hollow cylinder with heat generation, outside surface insulated

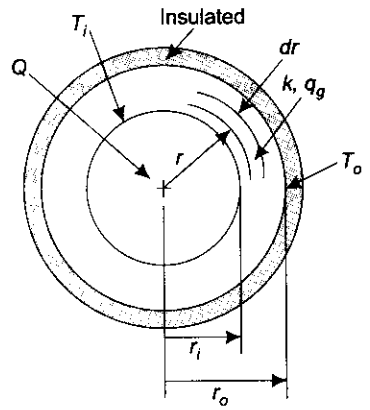

FIGURE 5.10 Hollow cylinder with heat generation, outside the surface insulated

Alternative Method:

See Fig. 5.10.

Consider any radius r in the cylindrical shell as shown.

Since the outside surface is insulated, heat generated within the volume between $r = r_o$ and $r = r$, must travel only inward; and, this heat must be equal to the heat conducted from the surface at radius r.

Writing this heat balance,

$$q_g \cdot \pi \cdot \left(r_o^2 - r^2\right) \cdot L = k \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{dT}{dr}$$

where, dT/dr is the temperature gradient at radius r.

Note that the term on the RHS has positive sign, since, now, the heat transfer is from outside to inside, i.e. in the negative r-direction (because the outside surface is insulated).

i.e.

$$dT = \frac{q_g}{2 \cdot k \cdot r}\left(r_o^2 - r^2\right) \cdot dr \qquad \text{...(a)}$$

Integrating:

$$T(r) = \frac{q_g \cdot r_o^2}{2 \cdot k} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k} + C \qquad \text{...(b)}$$

Eq. b is the general solution for temperature distribution. The integration constant C is obtained by the B.C.:

At $r = r_i$, $T = T_i$

Applying this B.C. to Eq. b:

$$C = T_i - \frac{q_g \cdot r_o^2}{2 \cdot k} \cdot \ln(r_i) + \frac{q_g \cdot r_i^2}{4 \cdot k}$$

Substituting value of C back in Eq. b:

$$T(r) = \frac{q_g \cdot r_o^2}{2 \cdot k} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k} + T_i - \frac{q_g \cdot r_o^2}{2 \cdot k} \cdot \ln(r_i) + \frac{q_g \cdot r_i^2}{4 \cdot k}$$

i.e.

$$T(r) = T_i + \frac{q_g \cdot r_o^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - \left(\frac{r}{r_o}\right)^2\right] \qquad \text{...(5.31)}$$

Eq. 5.31 gives the temperature distribution in a hollow cylinder with heat generation, insulated on the outside surface, in terms of the inner wall temperature, T_i.

Putting $r = r_o$ and $T = T_o$ in Eq. 5.31, we get,

$$T_o - T_i = \frac{q_g \cdot r_o^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r_o}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - 1\right] \qquad \text{...(5.32)}$$

Eq. 5.32 is important, since it gives the maximum temperature drop in the cylindrical shell, when there is internal heat generation and the outside surface is insulated.

If either of T_o or T_i is given in a problem, then the other temperature can be calculated using Eq. 5.32.

Convection boundary condition:

If heat is carried away at the inner surface by a fluid at a temperature T_a flowing on the surface with a convective heat transfer coefficient, h_a, then, it is the fluid temperature that is known and not the surface temperature, T_i. In such cases, we relate the surface temperature and fluid temperature by an energy balance at the surface, i.e. heat generated within the body and conducted to the inner surface is equal to the heat convected away by the fluid at the surface.

i.e.

$$q_g \cdot \pi \cdot (r_o^2 - r_i^2) \cdot L = h_a \cdot 2 \cdot \pi \cdot r_i \cdot L \cdot (T_i - T_a)$$

i.e.

$$T_i = T_a + \frac{q_g \cdot \left(r_o^2 - r_i^2\right)}{2 \cdot h_a \cdot r_i} \qquad \text{...(c)}$$

Using Eq. c in Eq. 5.31, we get:

$$T(r) = T_a + \frac{q_g \cdot \left(r_o^2 - r_i^2\right)}{2 \cdot h_a \cdot r_i} + \frac{q_g \cdot r_o^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - \left(\frac{r}{r_o}\right)^2\right] \qquad \text{...(5.33)}$$

Eq. 5.33 gives the temperature distribution in a hollow cylinder with heat generation, insulated on the outside surface, cooled by a fluid on the inside, in terms of the fluid temperature, T_a.

5.3.2.4 Analysis with variable thermal conductivity. In the above analysis, thermal conductivity of the material was assumed to be constant. Now, let us make an analysis when the thermal conductivity varies linearly with temperature as:

$$k(T) = k_o(1 + \beta T),$$

where, k_o and β are constants.

Again, considering Fig. 5.10, we have, from heat balance:

$$q_g \cdot \pi \cdot \left(r_o^2 - r^2\right) \cdot L = k(T) \cdot 2 \cdot \pi \cdot r \cdot L \cdot \frac{dT}{dr}$$

where, dT/dr is the temperature gradient at radius r.

Separating the variables and substituting for $k(T)$,

$$k_o \cdot (1 + \beta \cdot T) \cdot dT = \frac{q_g}{2 \cdot r}\left(r_o^2 - r^2\right) \cdot dr = \frac{q_g \cdot r_o^2}{2} \cdot \frac{dr}{r} - \frac{q_g}{2} \cdot r \cdot dr$$

Integrating:
$$T + \frac{\beta \cdot T^2}{2} = \frac{q_g \cdot r_o^2}{2 \cdot k_o} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k_o} + C \qquad \text{...(d)}$$

In Eq. d get the integration constant, C from the B.C.: at $r = r_o$, $T = T_o$

i.e.
$$C = T_o + \frac{\beta \cdot T_o^2}{2} - \frac{q_g \cdot r_o^2}{2 \cdot k_o} \cdot \ln(r_o) + \frac{q_g \cdot r_o^2}{4 \cdot k_o}$$

Substitute value of C back in Eq. d:

$$\frac{\beta \cdot T^2}{2} + T - \left(\frac{q_g \cdot r_o^2}{2 \cdot k_o} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k_o} + T_o + \frac{\beta \cdot T_o^2}{2} - \frac{q_g \cdot r_o^2}{2 \cdot k_o} \cdot \ln(r_o) + \frac{q_g \cdot r_o^2}{4 \cdot k_o}\right) = 0 \qquad \text{...(e)}$$

Eq. e is a quadratic in T. Its positive root is given by:

$$T(r) = \frac{-1 + \sqrt{1 + 4 \cdot \frac{\beta}{2} \cdot \left(\frac{q_g \cdot r_o^2}{2 \cdot k_o} \cdot \ln(r) - \frac{q_g \cdot r^2}{4 \cdot k_o} + T_o + \frac{\beta \cdot T_o^2}{2} - \frac{q_g \cdot r_o^2}{2 \cdot k_o} \cdot \ln(r_o) + \frac{q_g \cdot r_o^2}{4 \cdot k_o}\right)}}{2 \cdot \frac{\beta}{2}}$$

After some manipulation, we get:

$$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_o\right)^2 - \frac{q_g \cdot r_o^2}{2 \cdot \beta \cdot k_o} \cdot \left[2 \cdot \ln\left(\frac{r_o}{r}\right) - \left(\frac{r_o}{r}\right)^2 - 1\right]} \qquad \text{...(5.34)}$$

Eq. 5.34 gives the temperature distribution in a hollow cylinder with internal heat generation when the outside surface is insulated and the thermal conductivity varies linearly with temperature.

5.3.2.5 Hollow cylinder with both the surfaces maintained at constant temperatures. Consider steady state, one-dimensional heat transfer in a hollow cylinder of length L, inside radius r_i and outside radius r_o, with a uniform internal heat generation rate of q_g (W/m^3). Thermal conductivity, k is constant. Let the temperatures on the inside and outside surfaces be T_i and T_o, respectively. The cylinder is losing heat from both the surfaces.

See Fig. 5.11.

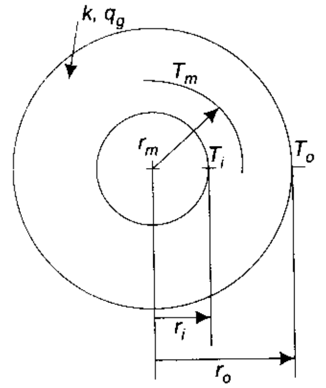

FIGURE 5.11 Hollow cylinder with heat generation, losing heat from both surfaces

Assumptions:

(i) Steady state conduction

(ii) One-dimensional conduction, in the r direction only

(iii) Homogeneous, isotropic material with constant k

(iv) Uniform internal heat generation rate, q_g (W/m^3).

With the above stipulations, the general differential equation in cylindrical coordinates (see Eq. 3.17 reduces to:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\cdot\frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

Integrating Eq. a twice, we get the general solution for temperature distribution, i.e. Eq. 5.18, as done in section 5.3.1:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \qquad \text{...(5.18)}$$

Eq. 5.18 is the general relation for temperature distribution along the radius, for a cylindrical system, with uniform heat generation. C_1 and C_2, the constants of integration are obtained by applying the boundary conditions.

In the present case, the B.C.'s are:

B.C.(i): at $r = r_i$ $T = T_i$, and

B.C.(ii): at $r = r_o$ $T = T_o$

Get C_1 and C_2 from these B.C.'s and substitute back in Eq. 5.18 to get the temperature distribution.

From B.C.(i) and Eq. 5.18:

$$T_i = \frac{-q_g \cdot r_i^2}{4 \cdot k} + C_1 \cdot \ln(r_i) + C_2 \qquad \text{...(a)}$$

From B.C.(ii) and Eq. 5.18:

$$T_o = \frac{-q_g \cdot r_o^2}{4 \cdot k} + C_1 \cdot \ln(r_o) + C_2 \qquad \text{...(b)}$$

Subtracting Eq. a from Eq. b:

$$T_o - T_i = \frac{q_g}{4 \cdot k}\cdot\left(r_i^2 - r_o^2\right) + C_1 \cdot \ln\left(\frac{r_o}{r_i}\right)$$

i.e.

$$C_1 = \frac{(T_o - T_i) + \frac{q_g}{4 \cdot k}\cdot\left(r_o^2 - r_i^2\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

And, from Eq. a:

$$C_2 = T_i + \frac{q_g \cdot r_i^2}{4 \cdot k} - \frac{(T_o - T_i) + \frac{q_g}{4 \cdot k}\cdot\left(r_o^2 - r_i^2\right)}{\ln\left(\frac{r_o}{r_i}\right)}\cdot \ln(r_i)$$

Temperature distribution in the cylindrical shell is obtained by substituting C_1 and C_2 in Eq. 5.18.

After lengthy algebraic manipulations, we get,

$$\frac{T(r)-T_i}{T_o-T_i} = \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} + \frac{q_g}{4 \cdot k} \cdot \frac{\left(r_o^2 - r_i^2\right)}{(T_o - T_i)} \cdot \left[\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} - \frac{\left(\frac{r}{r_i}\right)^2 - 1}{\left(\frac{r_o}{r_i}\right)^2 - 1}\right] \quad ....(5.35)$$

Position and value of maximum temperature:

Position of maximum temperature must lie somewhere between r_i and r_o, since heat is flowing to both inside and outside surfaces. Let the position be at a radius of r_m. Then, r_m is found out by differentiating the expression for $T(r)$ given by Eq. 5.35 w.r.t. r and equating to zero. Then, this value of r_m is substituted back in Eq. 5.35 to obtain T_{max}. The procedure will be illustrated in an example, later.

Heat transfer to both surfaces:

Knowing the temperature distribution, heat transfer rate is easily determined by applying the Fourier's law.

Heat transfer rate at the inner surface, $Q|_{r=r_i} = -k(2\pi r_i L)\,(dT/dr)|_{r=r_i}$

Heat transfer rate at the outer surface, $Q|_{r=r_o} = -k(2\pi r_o L)\,(dT/dr)|_{r=r_o}$

Note that heat transfer to inner surface will be negative since the heat flow is from outside to inside, i.e. in the negative r-direction.

Check: Sum of the amount of heats flowing to the inner and outer surfaces must be equal to the total amount of heat generated in the cylindrical shell.

Convective boundary conditions:

If heat is carried away at the inner surface by a fluid at a temperature T_a flowing on the surface with a convective heat transfer coefficient, h_a, and on the outer surface, by a fluid at a temperature T_b flowing on the surface with a convective heat transfer coefficient, h_b then, the surface temperatures can be related to the fluid temperatures by making an energy balance at the surfaces. i.e. heat generated within the body and conducted to the inner and outer surfaces must be equal to the heat convected away by the fluid at the respective surfaces.

See Example 5.10 for procedure of working out a numerical problem.

Alternative Method:

See Fig. 5.11.

Since heat is transferred from both the inside and outside surfaces, maximum temperature, T_m must occur somewhere in the shell. Let it occur at a radius r_m. Obviously, r_m lies in between r_i and r_o. Now, note that surface at r_m is an isothermal surface; also, since maximum temperature occurs at r_m, no heat will cross the surface at r_m i.e. dT/dr at $r = r_m$ will be zero. This also means that surface at r_m may be considered as representing an insulated boundary condition.

So, the cylindrical shell may be thought of as being made up of two shells; the inner shell, between $r = r_i$ and $r = r_m$, insulated on its 'outer periphery' and, an outer shell, between $r = r_m$ and $r = r_o$, insulated at its 'inner periphery'.

Then, maximum temperature difference for the inner shell and outer shell can be written from Eq. 5.32 and 5.28, respectively. So, we write:

For the 'inner shell' (insulated on the 'outer" surface):

$$T_m - T_i = \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r_m}{r_i}\right) + \left(\frac{r_i}{r_m}\right)^2 - 1\right] \quad ...(a)$$

Eq. a is obtained by replacing r_o by r_m and T_o by T_m in Eq. 5.32.

For the 'outer shell' (insulated on the 'inner' surface):

$$T_m - T_o = \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_m}\right) - 1\right] \quad ...(b)$$

Eq. b is obtained by replacing r_i by r_m and T_i by T_m in Eq. 5.28.

Subtracting Eq. a from b:

$$T_i - T_o = \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_m} \right) - 1 - 2 \cdot \ln\left(\frac{r_m}{r_i} \right) - \left(\frac{r_i}{r_m} \right)^2 + 1 \right]$$

i.e. $$T_i - T_o = \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m} \right)^2 - \left(\frac{r_i}{r_m} \right)^2 + 2 \cdot \ln\left(\frac{r_m}{r_o} \right) - 2 \cdot \ln\left(\frac{r_m}{r_i} \right) \right] \quad \text{...(c)}$$

Eq. c must be solved for r_m. We get:

$$T_i - T_o = \frac{q_g}{4 \cdot k} \cdot \left(r_o^2 - r_i^2 \right) + \frac{q_g \cdot r_m^2}{4 \cdot k} 2 \cdot \ln\left(\frac{r_m}{r_o} \cdot \frac{r_i}{r_m} \right)$$

i.e. $$T_i - T_o = \frac{q_g}{4 \cdot k} \cdot \left(r_o^2 - r_i^2 \right) + \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot 2 \cdot \ln\left(\frac{r_i}{r_o} \right)$$

i.e. $$T_i - T_o = \frac{q_g}{4 \cdot k} \cdot \left[\left(r_o^2 - r_i^2 \right) + 2 r_m^2 \cdot \ln\left(\frac{r_i}{r_o} \right) \right]$$

i.e. $$\frac{(T_i - T_o) \cdot 4 \cdot k}{q_g} = \left[\left(r_o^2 - r_i^2 \right) + 2 \cdot r_m^2 \cdot \ln\left(\frac{r_i}{r_o} \right) \right]$$

i.e. $$r_m^2 = \frac{(T_i - T_o) \cdot 4 \cdot k}{q_g \cdot 2 \cdot \ln\left(\frac{r_i}{r_o} \right)} - \frac{\left(r_o^2 - r_i^2 \right)}{2 \cdot \ln\left(\frac{r_i}{r_o} \right)}$$

i.e. $$r_m^2 = \frac{q_g \cdot \left(r_o^2 - r_i^2 \right) - 4 \cdot k \cdot (T_i - T_o)}{q_g \cdot 2 \cdot \ln\left(\frac{r_o}{r_i} \right)}$$

i.e. $$r_m = \sqrt{\frac{q_g \cdot \left(r_o^2 - r_i^2 \right) - 4 \cdot k \cdot (T_i - T_o)}{q_g \cdot 2 \cdot \ln\left(\frac{r_o}{r_i} \right)}} \quad \text{...(5.36)}$$

Substituting the value of r_m from Eq. 5.36 in either of Eqs. a or b, we get the maximum temperature in the shell.

Then, temperature distribution in the inner shell is determined from Eq. 5.32 and that in the outer shell is determined from Eq. 5.28.

When T_i and T_o are equal:

When the cooling on the surfaces is such that both T_i and T_o are the same, an interesting situation develops: then, it is seen from Eq. 5.36 that, position of maximum temperature in the shell is given by:

$$r_m = \sqrt{\frac{r_o^2 - r_i^2}{2 \cdot \ln\left(\frac{r_o}{r_i} \right)}}$$

i.e. r_m depends only on the physical dimensions of the cylindrical shell and not on the thermal conditions.

For example, for a hollow cylinder with $r_i = 5$ cm and $r_o = 10$ cm, when the inside and outside surfaces are maintained at the same temperature ($T_i = T_o$), the maximum temperature in the shell occurs at a radius of:

$$r_m = \sqrt{\frac{10^2 - 5^2}{2 \cdot \ln\left(\frac{10}{5}\right)}}$$

i.e. $$r_m = 7.355 \text{ cm.}$$

This result is valid, whatever may be the value of uniform heat generation.

Example 5.10. A hollow cylinder 6 cm ID, 9 cm OD, has a heat generation rate of 5×10^6 W/m^3. Inner surface is maintained at 450°C and outer surface at 350°C. k of the material is 3 W/(mK).

(i) Determine the location and value of maximum temperature.
(ii) What is the temperature at mid-thickness of the shell?
(iii) Determine the fraction of heat generated going to the inner surface, and
(iv) Sketch the temperature profile.

Solution. See Figure Example 5.10.

Data:

$r_i := 0.03$ m $r_o := 0.045$ m $L := 1$ m

$T_i := 450$°C $T_o := 350$°C $k := 3$ W/(mK)

$q_g := 5 \times 10^6$ W/m^3

Position of maximum temperature can be immediately determined from Eq. 5.36 and then, the value of maximum temperature may be determined from Eq. 5.28 or 5.32.

However, let us work out this problem from first principles and then, verify the results from the formulas already derived.

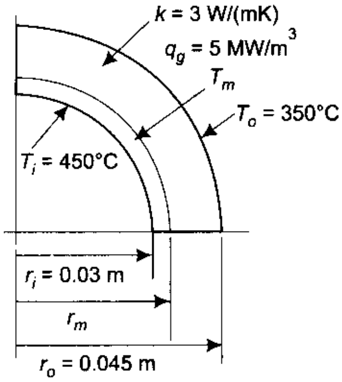

FIGURE Example 5.10 Hollow cylinder with heat generation, losing heat from both surfaces

Temperature distribution:

For the assumption of one-dimensional, steady state conduction with heat generation in a cylindrical geometry, we have the governing differential equation:

$$\frac{d^2T}{dr^2} + \frac{1}{r} \cdot \frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

Multiplying by r: $$r \cdot \frac{d^2T}{dr^2} + \frac{dT}{dr} + \frac{q_g \cdot r}{k} = 0$$

i.e. $$\frac{d}{dr}\left(r \cdot \frac{dT}{dr}\right) = \frac{-q_g \cdot r}{k}$$

Integrating: $$r \cdot \frac{dT}{dr} = \frac{-q_g \cdot r^2}{2 \cdot k} + C_1$$

i.e. $$\frac{dT}{dr} = \frac{-q_g \cdot r}{2 \cdot k} + \frac{C_1}{r} \qquad \text{...(b)}$$

Integrating again: $$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \qquad \text{...(5.18)}$$

Eq. 5.18 gives the temperature distribution. C_1 and C_2 are determined from the B.C.'s:

B.C.(i): at $r = r_i$, we have $T = T_i$

B.C.(ii): $r = r_o$, we have $T = T_o$

From B.C.(i) and Eq. 5.18:

$$T_i = \frac{-q_g \cdot r_i^2}{4 \cdot k} + C_1 \cdot \ln(r_i) + C_2 \qquad \text{...(c)}$$

From B.C.(ii) and Eq. 5.18:

$$T_o = \frac{-q_g \cdot r_o^2}{4 \cdot k} + C_1 \cdot \ln(r_o) + C_2 \qquad \text{...(d)}$$

Subtracting Eq. c from Eq. d:

$$T_o - T_i = \frac{q_g}{4 \cdot k} \cdot (r_i^2 - r_o^2) + C_1 \cdot \ln\left(\frac{r_o}{r_i}\right)$$

i.e. $$C_1 := \frac{(T_o - T_i) + \frac{q_g}{4 \cdot k} \cdot (r_o^2 - r_i^2)}{\ln\left(\frac{r_o}{r_i}\right)}$$ *(define integration constant C_1)*

i.e. $C_1 = 909.449$ *(value of C_1, after substituting numerical values from data)*

and, from Eq. c:

$$C_2 := T_i + \frac{q_g \cdot r_i^2}{4 \cdot k} - \frac{(T_o - T_i) + \frac{q_g}{4 \cdot k} \cdot (r_o^2 - r_i^2)}{\ln\left(\frac{r_o}{r_i}\right)} \cdot \ln(r_i)$$ *(define integration constant C_2)*

i.e. $C_2 = 4.01404 \times 10^3$ *(value of C_2, after substituting numerical vaues from data)*

Substituting C_1 and C_2 in Eq. 5.18, we get the temperature distribution as:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2$$

i.e. $$T(r) := -4.16667 \times 10^5 \cdot r^2 + 909.449 \ln(r) + 4.01404 \times 10^3 \qquad \text{...(e)}$$

Eq. e is the desired temperature distribution in the shell as a function of radius, r.

Position and value of maximum temperature:

To get the position where maximum temperature occurs, differentiate Eq. e w.r.t. r and equate to zero. Let the location be at a radius r_m. Then, substitute r_m back in Eq. e to get value of maximum temperature, T_m.

Differentiating Eq. e w.r.t. r and equating to zero:

$$\frac{d}{dr}T(r) = -4.16667 \times 10^5 \cdot 2 \cdot r + \frac{909.449}{r} = 0$$

i.e. $$r_m = \sqrt{\frac{909.449}{2 \cdot 4.16667 \times 10^5}}$$

i.e. $r_m = 0.033$ m *(position of maximum temperature)*

And, substituting r_m in Eq. e, we get T_{max}

$T(r_m) = 457.935°C$ ***(value of maximum temperature)***

Note: In Mathcad, there is no need to actually differentiate Eq. e and equate to zero, then solve etc. First, define $T'(r)$ as the first derivative of $T(r) = dT(r)/dr$ and then use the solve block to get the root of $T'(r) = 0$. For doing this, assume a trial value of r to start with. Procedure is shown below:

$$T'(r) := \frac{d}{dr}T(r)$$ *(define first derivative of $T(r)$)*

$r := 0.03$ m *(trial value of r)*

Given

$T'(r) = 0$

$r_{max} :=$ Find (r) *(define r_{max})*

i.e. $r_{max} = 0.033$ m *(position of maximum temperature...**verified.**)*

Check: check also from the direct formula Eq. (5.36) for r_m:

$$r_m := \sqrt{\frac{q_g \cdot (r_o^2 - r_i^2) - 4 \cdot k\,(T_i - T_o)}{q_g \cdot 2 \cdot \ln\left(\frac{r_o}{r_i}\right)}} \qquad ...(5.36)$$

i.e. $r_m = 0.033$ m *(position of maximum temperature...checks.)*

Check the maximum temperature from Eq. 5.28:

We have, for a shell insulated on the inner surface:

$$T_i - T_o = \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i}\right) - 1 \right] \qquad ...(5.28)$$

Apply this formula for the 'outer shell', i.e. between $r = r_m$ and $r = r_o$,: Now, replacing T_i by the maximum temperature T_m and r_i by r_m, we get:

$$T_m - T_o = \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_m}\right) - 1 \right]$$

i.e. $$T_m := T_o + \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_m}\right) - 1 \right]$$ *(define T_m, the maximum temperature)*

i.e. $T_m = 457.931$°C *(value of maximum temperature...checks.)*

Temperature at mid-thickness i.e. at $r = 0.0375$ m:

Substitute $r = 0.0375$ in Eq. e for $T(r)$:

i.e. $T(0.0375) = 442.004$°C ***(temperature at mid-thickness of shell)***

Fraction of heat generated going to inner surface:

First, find the total heat generated in the shell:

$Q_{tot} := \pi \cdot (r_o^2 - r_i^2) \cdot L \cdot q_g$, W *...define the total heat generation in the shell = (Volume × q_g)*

i.e. $Q_{tot} = 1.767 \times 10^4$ W *(total heat generated in the shell)*

Heat going to inner surface is equal to the amount of heat generated between $r = r_i$ and $r = r_m$, since no heat crosses the isothermal surface at r_m.

$Q_{inner} := \pi \cdot (r_m^2 - r_i^2) \cdot L \cdot q_g$ W *(define the heat going to inner surface of the shell)*

i.e. $Q_{inner} = 3.006 \times 10^3$ W *(heat going to inner surface of the shell)*

Therefore, fraction of heat going to inner surface:

$$\text{Fraction} := \frac{Q_{inner}}{Q_{tot}}$$ *...define Fraction*

i.e. Fraction = 0.17 *(i.e. **17% of the total heat generated goes to the inner surface.**)*

Note: Heat removed at the inner surface can also be found out by applying the Fourier's law at $r = r_i$: Remember, temperature gradient is given by $T'(r)$.

$Q_{inner} := -k \cdot (2 \cdot \pi \cdot r_i \cdot L) \cdot T'(r_i)$ W *(define heat flow at inner surface...Fourier's law)*

i.e. $Q_{inner} = -3.006 \times 10^3$ W *(negative sign indicates that heat flow is radially inwards...**verified.**)*

To sketch the temperature distribution:

To sketch the temperature profile in the shell, define a range variable r, varying from 0.03 to 0.045 m, with an increment of 0.001 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.10(b).

$r := 0.03, 0.031, ..., 0.045$ *(define a range variable r..starting value = 0.03, next value = 0.031 m and last value = 0.045 m)*

Note from the graph that maximum temperature occurs at $r = 0.033$ m.

Example 5.11. A high temperature, gas cooled nuclear reactor consists of a composite cylindrical wall for which a thorium fuel element ($k = 57$ W/(mK)) is encased in graphite ($k = 3$ W/(mK)) and gaseous helium flows through an

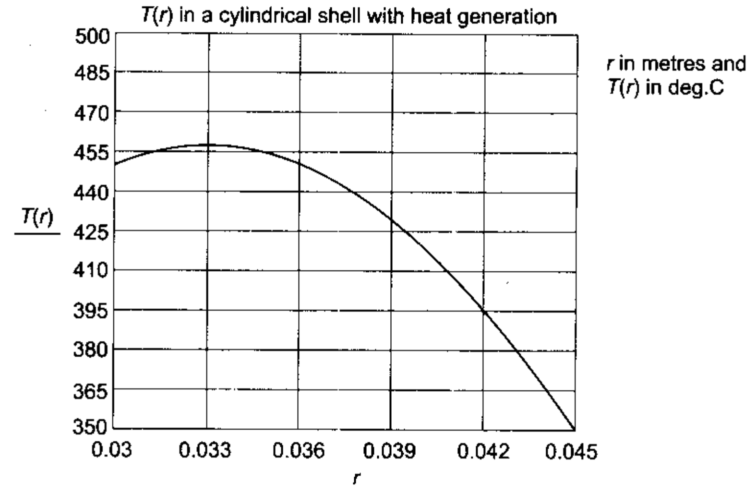

FIGURE Example 5.10(b)

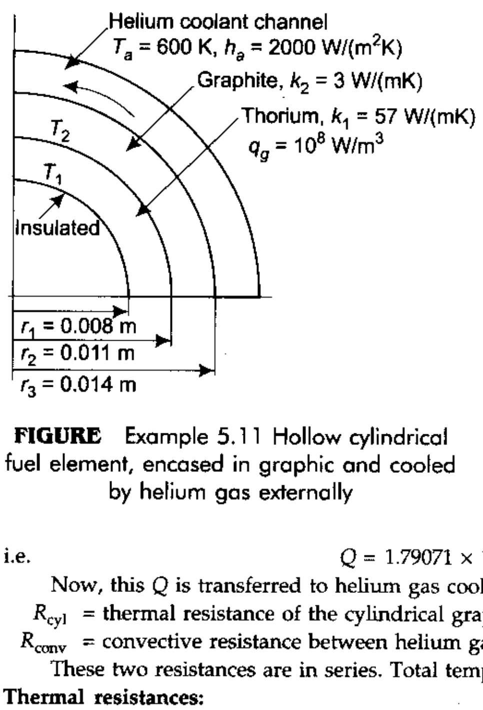

FIGURE Example 5.11 Hollow cylindrical fuel element, encased in graphic and cooled by helium gas externally

annular coolant channel, as shown in Fig. 5.23. Consider conditions for which helium temp T_a = 600 K and convective coefficient h_a at the outer surface of graphite = 2000 W/(m^{2}K). If r_1 = 8 mm, r_2 = 11 mm, r_3 = 14 mm and $q_g = 10^8$ W/m^3, find out temperatures T_1 and T_2, i.e. at inner and outer surfaces of the fuel element. Also, draw the temperature profile in the fuel element and graphite.

Solution. See Figure Example 5.11.

Data:

$r_1 := 0.008$ m $\quad r_2 := 0.011$ m $\quad r_3 := 0.014$ m

$q_g := 10^8$ W/m^3 $\quad k_1 := 57$ W/(mK) $\quad k_2 := 3$ W/(mK)

$T_a := 600$ K $\quad h_a := 2000$ W/(m^{2}K) $\quad L := 1$ m

Find out T_1 and T_2.

Note that heat generation is only in thorium. Inside surface of thorium is insulated. So, in steady state, all the heat generated in thorium flows out by conduction through graphite shell and then by convection to helium gas.

Total heat generation rate, Q:

$Q := q_g \cdot \text{Volume}$

i.e. $Q = q_g \cdot \pi \cdot \left(r_2^2 - r_1^2\right) \cdot L$, W *(define q, total heat generated)*

i.e. $Q = 1.79071 \times 10^4$ W *(total heat generated in thorium fuel element)*

Now, this Q is transferred to helium gas coolant. Thermal resistances involved are:

R_{cyl} = thermal resistance of the cylindrical graphite shell, and

R_{conv} = convective resistance between helium gas and outer surface of graphite shell.

These two resistances are in series. Total temperature potential, $\Delta T = (T_2 - T_a)$

Thermal resistances:

$$R_{cyl} := \frac{\ln\left(\frac{r_3}{r_2}\right)}{2 \cdot \pi \cdot k_2 \cdot L} \text{ C/W}$$ *(define thermal resistance of graphite shell)*

i.e. $R_{cyl} = 0.013$ C/W *(thermal resistance of graphite shell)*

$$R_{conv} := \frac{1}{h_a \cdot 2 \cdot \pi \cdot r_3 \cdot L} \text{ C/W}$$ *(define convective resistance on outside surface of graphite shell)*

i.e. $R_{conv} = 5.684 \times 10^{-3}$ C/W *(convective resistance on outside surface of graphite shell)*

Now, we have: $$Q = \frac{T_2 - T_a}{R_{cyl} + R_{conv}}$$

Therefore, $T_2 := Q \cdot (R_{cyl} + R_{conv}) + T_a$ K *(define T_2)*

i.e. $T_2 = 930.89$ K ***(temperature at outer surface of fuel element.)***

To get temperature profile in thorium and then, find T_1:

T_1 is the temperature at the inner surface of the fuel element. It is insulated on the inner surface. So, we can apply Eq. 5.27, derived earlier for a hollow cylinder with heat generation, with the inner surface insulated. However, first we will solve it from first principles and then verify the results from Eq. 5.27:

For the assumption of one-dimensional, steady state conduction with heat generation in a cylindrical geometry, we have the governing differential equation:

$$\frac{d^2T}{dr^2} + \frac{1}{r} \cdot \frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad \text{...(a)}$$

Multiplying by r: $$r \cdot \frac{d^2T}{dr^2} + \frac{dT}{dr} + \frac{q_g \cdot r}{k} = 0$$

i.e. $$\frac{d}{dr}\left(r \cdot \frac{dT}{dr}\right) = \frac{-q_g \cdot r}{k}$$

Integrating: $$r \cdot \frac{dT}{dr} = \frac{-q_g \cdot r^2}{2 \cdot k} + C_1$$

i.e. $$\frac{dT}{dr} = \frac{-q_g \cdot r}{2 \cdot k} + \frac{C_1}{r} \qquad \text{...(b)}$$

Integrating again: $$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \qquad \text{...(5.18)}$$

Eq. 5.18 gives the temperature distribution. C_1 and C_2 are determined from the B.C.'s:

B.C.(i): at $r = r_1$, we have $dT/dr = 0$, since inner surface is insulated.

B.C.(ii): at $r = r_2$, we have $T = T_2$,

From B.C.(i) and Eq. b

$$0 = \frac{-q_g \cdot r_1}{2 \cdot k_1} + \frac{C_1}{r_1}$$

i.e. $$C_1 := \frac{+q_g \cdot r_1^2}{2 \cdot k_1}$$ *(define C_1)*

i.e. $C_1 = 56.14035$ *(value of integration constant C_1)*

From B.C.(ii) and Eq. 5.18

$$T_2 = \frac{-q_g \cdot r_2^2}{4 \cdot k_1} + C_1 \cdot \ln(r_2) + C_2$$

i.e. $$C_2 := T_2 + \frac{-q_g \cdot r_2^2}{4 \cdot k_1} - C_1 \cdot \ln(r_2)$$ *(define C_2)*

i.e. $C_2 = 1.23714 \times 10^3$ *(value of integration constant C_2)*

Substituting C_1 and C_2 in Eq. 5.18,

$$T(r) := -4.38596 \times 10^5 \cdot r^2 + 56.14035 \ln(r) + 1237.14 \qquad \text{...(c)...defines } T(r)$$

Eq. c gives temperature profile in the thorium fuel element.

Temperature T_1 on the inner surface of thorium:

Put $r = 0.008$ m in Eq. c

$T(0.008) = 938.007$

i.e. $T_1 = 938.007$ K ***(temperature on the inner surface of thorium fuel element.)***

Verify from Eq. 5.27:
We have:

$$T(r) = T_o + \frac{+q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r}\right) - \left(\frac{r}{r_i}\right)^2 \right] \quad ...(5.27)$$

i.e. $$T(r) := T_2 + \frac{+q_g \cdot r_1^2}{4 \cdot k_1} \cdot \left[\left(\frac{r_2}{r_1}\right)^2 - 2 \cdot \ln\left(\frac{r_2}{r}\right) - \left(\frac{r}{r_1}\right)^2 \right]$$ *(Eq. 5.27, with notations of this problem)*

i.e. $T(0.008) = 938.012$ K *(**temperature at the inner surface of the fuel element...verified.**)*

Check:

Heat flux at the interface must be the same for thorium as well as graphite:

i.e. $$-k \cdot \frac{dT(r)}{dr} = -k \cdot \frac{dt(r_s)}{dr}$$

at $r = r_s = 0.011$ m, where $T(r)$ is temperature profile for thorium and $t(r_s)$ is temperature profile for graphite.

Temperature profile for thorium is already obtained as:

$$T(r) = -4.38596 \times 10^5 \cdot r^2 + 56.14035 \ln(r) + 1237.14$$

Temperature profile for graphite is, from Eq. 4.34 for a cylindrical shell:

$$t(r_s) = T_2 + \frac{T_3 - T_2}{\ln\left(\frac{r_3}{r_2}\right)} \cdot \ln\left(\frac{r_s}{r_2}\right)$$ *(Eq. 4.34, with notations of this problem)*

where, r_s = any radius within the graphite shell

Now, define their first derivatives w.r.t. radius:

$$T'(r) := \frac{d}{dr} T(r) \text{ and,}$$

$$t'(r_s) := \frac{d}{dr} t(r_s)$$

Therefore, heat flux in thorium at r = 0.011 m:

$$q_{thorium} = -k_1 \cdot T'(0.011)$$ *...define heat flux, from Fourier's law*

i.e. $$q_{thorium} = 2.591 \times 10^5 \text{ W/m}^2$$ *...heat flux in thorium at the interface*

Heat flux in graphite at $r = 0.011$ m:

$$q_{graphite} = -k_2 \cdot t'(0.011)$$ *...define heat flux, from Fourier's law*

i.e. $$q_{graphite} = 2.591 \times 10^5 \text{ W/m}^2$$ *...heat flux in graphite at the interface*

Therefore, we observe that heat fluxes are same for thorium and graphite at the interface

(verified.)

To sketch the temperature profiles:

For temperature profile in graphite, we use Eq. 4.34, which was derived for a cylindrical shell with no heat generation.

$$t(r_s) = T_2 + \frac{T_3 - T_2}{\ln\left(\frac{r_3}{r_2}\right)} \cdot \ln\left(\frac{r_s}{r_2}\right)$$ *...4.34, with notations of this problem*

where, r_s = any radius within the graphite shell.

To sketch the temperature profile in the thorium fuel element, define a range variable r, varying from 0.008 to 0.011 m, with an increment of 0.0001 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears.

$r := 0.008, 0.0081, \ldots, 0.011$ *(define a range variable r..starting value = 0.008, next value = 0.0081 m and last value = 0.011 m)*

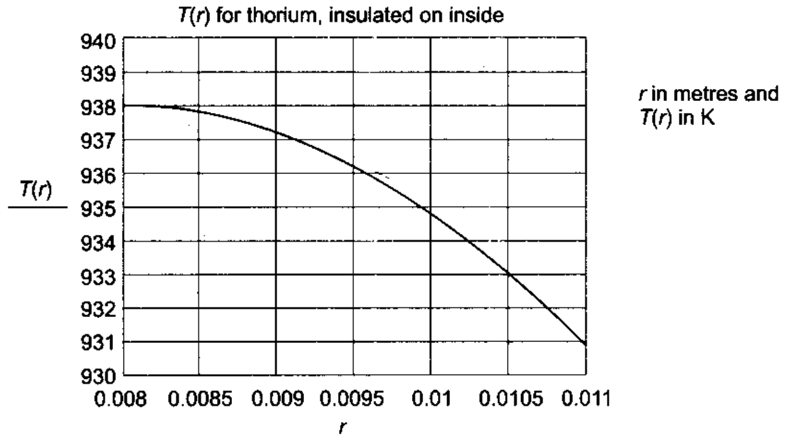

FIGURE Example 5.11(b)

It may be seen from the graph that at r = 0.008 m, temperature T_1 is 938.01 K and, at r = 0.011 m temperature T_2 is 930.9 K.

Similarly, **to sketch the temperature profile in the graphite shell**, define a range variable r_s, varying from 0.011 to 0.014 m, with an increment of 0.00001 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r_s and t(r_s), respectively. Click anywhere outside the graph region, and immediately the graph appears. Fig. Ex. 5.11(c)

r_s := 0.011, 0.01101, ..., 0.014 *(define a range variable r..starting value = 0.011, next value = 0.01101 m and last value = 0.014 m)*

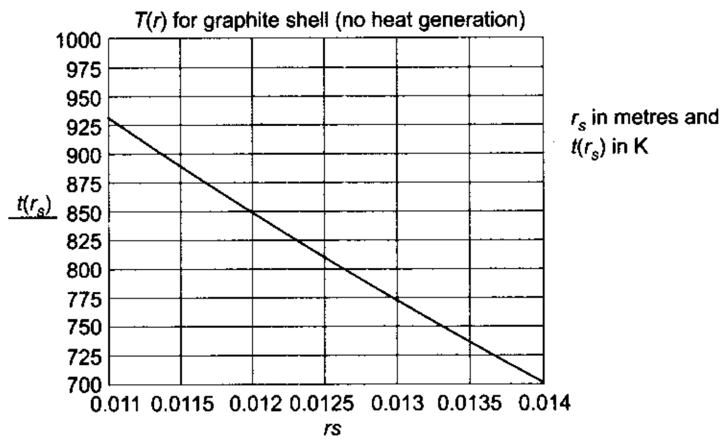

FIGURE Example 5.11(c)

It may be seen from the graph that at at r = 0.011 m, temperature T_2 is 930.9 K, and at r = 0.014 m, temperature T_3 is 701.8 K.

Example 5.12. A hollow conductor with r_i = 0.6 cm, r_o = 0.8 cm is made up of metal of k = 20 W/(mK) and electrical resistance per metre of 0.03 ohms. Find the maximum allowable current if the temperature is not to exceed 50°C anywhere in the conductor. The cooling fluid inside is at 38°C. (Conductor is insulated on the outside).

Solution. See Figure Example 5.12.

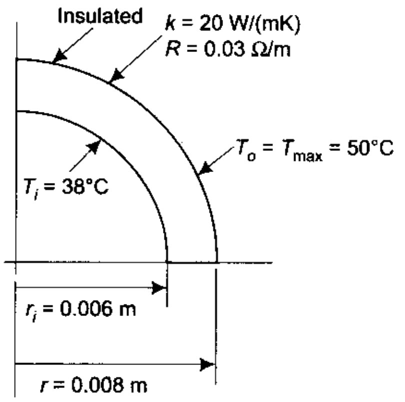

FIGURE Example 5.12 Hollow conductor with heat generation, insulated on outside surface, cooled on inside

Data:

r_i := 0.006 m r_o := 0.008 m k := 20 W/mK

R := 0.03 Ohms/m length T_i := 38°C L := 1 m

Note: temperature on inside surface is assumed as that of the fluid flowing, since heat transfer coefficient between the surface and the fluid is not given.

T_o := 50°C *(temperature of outer surface ...**this is also maximum temperature in conductor, since conductor is insulated on outside**)*

This is the case of a hollow cylinder with heat generation, cooled from inside and insulated on the outside surface. Both the inside and outside temperatures are known. So, one can use Eq. 5.32 directly to get q_g. Once q_g is known, the current, I can easily be calculated.

We shall, however, solve the problem from first principles and then check the result with Eq. 5.32:

As shown in the earlier two examples, the general equation for temperature distribution in a hollow cylinder with heat generation is given by Eq. 5.18, i.e.

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \qquad \text{...(5.18)}$$

Eq. 5.18 gives the temperature distribution. C_1 and C_2 are determined from the B.C.'s:

B.C.(i) at $r = r_i$, we have $T = T_i$, known temperature

B.C.(ii) at $r = r_o$, we have: $dT/dr = 0$, since insulated.

From 5.18, we have:

$$\frac{dT}{dr} = \frac{-q_g \cdot r}{2 \cdot k} + \frac{C_1}{r} \qquad \text{...(a)}$$

From B.C.(ii) and Eq. a:

$$0 = \frac{-q_g \cdot r_o}{2 \cdot k} + \frac{C_1}{r_o}$$

i.e.

$$C_1 = \frac{q_g \cdot r_o^2}{2 \cdot k} \qquad \textit{(integration constant } C_1\textit{)}$$

From B.C.(i) and Eq. 5.18:

$$T_i = \frac{-q_g \cdot r_i^2}{4 \cdot k} + \frac{q_g \cdot r_o^2}{2 \cdot k} \cdot \ln(r_i) + C_2$$

or,

$$C_2 = T_i - \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[2 \cdot \left(\frac{r_o}{r_i}\right)^2 \cdot \ln(r_i) - 1\right]$$

Substituting C_1 and C_2 in Eq. 5.18:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + \frac{q_g \cdot r_o^2}{2 \cdot k} \cdot \ln(r) + T_i - \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[2 \cdot \left(\frac{r_o}{r_i}\right)^2 \cdot \ln(r_i) - 1\right]$$

i.e.

$$T(r) = \frac{-q_g \cdot r^2}{80} + q_g (1.6 \times 10^{-6}) \cdot \ln(r) + 38 + 8.63559 \times 10^{-6} \cdot q_g \qquad \text{...(b)}$$

Eq. b is the desired expression for temperature distribution in a hollow cylinder with heat generation, when the outer surface is insulated.

Now, by data: at r = 0.008 m (i.e. at $r = r_0$), T = 50°C

Put this in Eq. b and solve to get q_g:

i.e. $$50 = \frac{-q_g \cdot (0.008)^2}{80} + q_g \cdot (1.6 \times 10^{-6}) \cdot \ln(0.008) + 38 + 8.63559 \times 10^{-6} \cdot q_g$$

i.e. $$q_g := \frac{50 - 38}{(1.6 \times 10^{-6}) \cdot \ln(0.008) + 8.63559 \times 10^{-6} - \frac{(0.008)^2}{80}} \text{ W/m}^3 \qquad \textit{(define } q_g\textit{)}$$

i.e. $q_g = 1.088 \times 10^8$ W/m^3 ***...heat generation rate***

Verify: Verify this result using Eq. 5.32, already derived, for a hollow cylinder with heat generation when the outer surface is insulated.

$$T_o - T_1 = \frac{q_g \cdot r_o^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r_o}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - 1 \right] \qquad \text{...(5.32)}$$

Therefore $$q_g := \frac{4 \cdot k \cdot (T_o - T_i)}{r_o^2 \cdot \left[2 \cdot \ln\left(\frac{r_o}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - 1 \right]} \text{ W/m}^3 \qquad \textit{(define } q_g\textit{)}$$

i.e. $q_g = 1.088 \times 10^8$ W/m^3 ***(heat generation rate...verified.)***

Maximum allowable current in conductor:

Let the current be I (A). Then,

$$q_g = \frac{Q_{gen}}{\text{Volume}} = \frac{I^2 \cdot R}{\pi \cdot (r_o^2 - r_i^2) \cdot L}$$

Therefore, $$I := \sqrt{\frac{q_g \cdot \pi \cdot (r_o^2 - r_i^2) \cdot L}{R}} \text{ A} \qquad \textit{(define I)}$$

i.e. $I = 564.824$ A ***(maximum allowable current in conductor)***

For completeness, **let us draw the temperature profile too:**

To sketch the temperature profile in the shell, define a range variable r, varying from 0.006 to 0.008 m, with an increment of 0.0001 m. Then, choose x-y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.12(b).

$$T(r) := \left[\frac{-q_g \cdot r^2}{80} + q_g \cdot (1.6 \times 10^{-6}) \cdot \ln(r) + 38 + 8.63559 \times 10^{-6} \cdot q_g \right] \qquad \textit{...define } T(r)\text{...(b)}$$

$r := 0.006, 0.0061, \ldots, 0.008$ *(define a range variable r..starting value = 0.006, next value = 0.0061 m and last value = 0.008 m)*

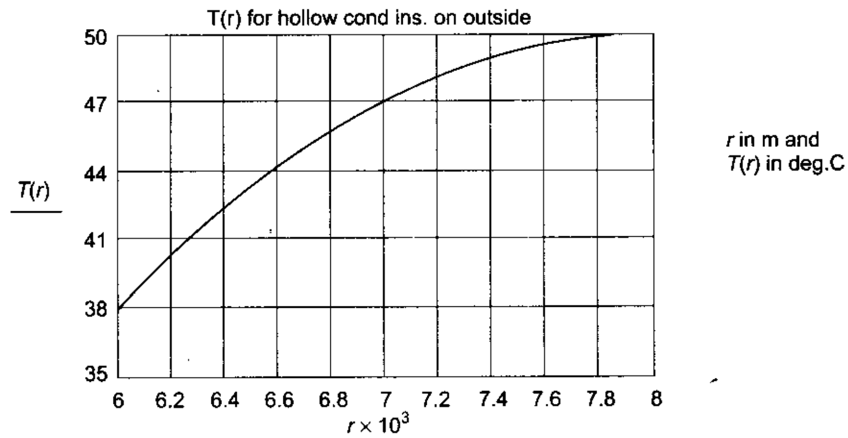

FIGURE Example 5.12(b)

It may be verified from the graph that on the inside and outside surfaces, the temperature are 38°C and 50°C, respectively.

Note: In Examples 5.10, 5.11 and 5.12, we have solved from first principles, problems with all the three types of B.C.'s, namely, cylindrical shell losing heat from both the surfaces, cylindrical shell insulated on the inside surface, and cylindrical shell insulated on the outside surface. Temperature profiles are also drawn for all the three cases. Student is advised to study the procedure followed carefully.

Example 5.13. A long, hollow cylinder has inner and outer radii as 5 cm and 15 cm, respectively. It generates heat at the rate of 1.0 kW/m^3. If the maximum temperature occurs at the radius 10 cm and the temperature of the outer surface is 50°C, find:

(i) temperature of the inner surface

(ii) maximum temperature in the cylinder.

Assume k = 0.5 W/(mC).

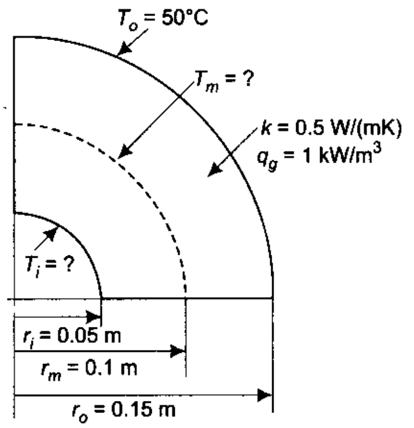

FIGURE Example 5.13 Hollow conductor with heat generation, losing heat on both surfaces, location of maximum temperature given

Soution. See Figure Example 5.13.

Data:

$r_i := 0.05$ m $\quad r_o := 0.15$ m $\quad r_m := 0.1$ m

$L := 1$ m $\quad T_o := 50$°C

$k := 0.5$ W/(mK) $\quad q_g := 1000$ W/m^3

T_i, temperature of inner surface is to be found out; also, the value of maximum temperature, T_m:

This is a problem of cylindrical shell with heat generation and losing heat from both surfaces; position of maximum temperature is at a radius of 10 cm and it is equivalent to insulated surface since no heat crosses the position of maximum temperature So, the whole shell between the radii of 5 cm and 15 cm may be thought of as being made of two shells: one inner shell, between radii of 5 cm and 10 cm, insulated on the outer surface, and the other, an outer shell, between the radii of 10 cm and 15 cm, insulated on the inner surface.

To find heat generation rate, q_g:

We have, for a shell insulated on the inner surface:

$$T_i - T_o = \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i} \right) - 1 \right] \qquad \text{...(5.28)}$$

Apply this formula for the 'outer shell', i.e. between $r = r_m$ and $r = r_o$,: Now, replacing T_i by the maximum temperature T_m and r_i by r_m, we get:

$$T_m - T_o = \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_m} \right) - 1 \right]$$

i.e.

$$T_m := T_o + \frac{q_g \cdot r_m^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_m} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_m} \right) - 1 \right]$$ *(define T_m, the maximum temperature)*

i.e. $T_m = 52.195$°C *(value of maximum temperature)*

Temperature at the inner surface, T_i:

This is easily determined from Eq. 5.36, i.e.

$$r_m = \sqrt{\frac{q_g \cdot (r_o^2 - r_i^2) - 4 \cdot k \cdot (T_i - T_o)}{q_g \cdot 2 \cdot \ln\left(\frac{r_o}{r_i} \right)}} \qquad \text{...(5.36)}$$

Therefore,

$$T_i := \frac{q_g \cdot (r_o^2 - r_i^2) - r_m^2 \cdot q_g \cdot 2 \cdot \ln\left(\frac{r_o}{r_i}\right)}{4 \cdot k} + T_o \qquad \textit{(define } T_i\textit{)}$$

i.e. $T_i = 49.014°C$ ***(temperature on the inner surface)***

Alternatively:

Instead of applying direct formulas 5.28 and 5.36, which are rather complicated, it is, perhaps, easier to work from first principles:

As explained earlier, the general Eq. for temperature distribution in a cylindrical shell with heat generation is given by Eq. 5.18:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2 \qquad ...(5.18)$$

and,

$$\frac{dT(r)}{dr} = \frac{-q_g \cdot r}{2 \cdot k} + \frac{C_1}{r} \qquad ...(a)$$

Eq. 5.18 gives the temperature distribution. C_1 and C_2 are determined from the B.C.'s:

Position of maximum temperature is given as at a radius of 10 cm, i.e. at the radius of 10 cm, we have an isothermal surface and no heat crosses this surface i.e. it is equivalent to the B.C. $dT/dr = 0$ at $r = 10$ cm.

B.C.(i) at $r = 0.015$ m, we have: $T = 50°C$

B.C.(ii) at $r = 0.01$ m, we have $dT/dr = 0$

Applying these two B.C.'s, we get C_1 and C_2; then substitute C_1 and C_2 back in Eq. 5.18 to get the temperature profile. Then, maximum temperature is found out by simply putting $r = 0.1$ m in the equation for temperature profile.

From B.C.(ii) and Eq. a:

$$C_1 := \frac{q_g \cdot r_m^2}{2 \cdot k} \qquad \textit{(define } C_1\textit{)}$$

i.e. $C_1 = 10$ *(value of C_1, integration constant)*

From B.C.(i) and Eq. 5.18:

$$50 = \frac{-q_g \cdot r_o^2}{4 \cdot k} + C_1 \cdot \ln(r_o) + C_2$$

i.e.

$$C_2 := 50 + \frac{q_g \cdot r_o^2}{4 \cdot k} - C_1 \cdot \ln(r_o) \qquad \textit{(define } C_2\textit{)}$$

i.e. $C_2 = 80.2212$ *(C_2...integration constant)*

Therefore, temperature distribution is given by:

$$T(r) = \frac{-q_g \cdot r^2}{4 \cdot k} + C_1 \cdot \ln(r) + C_2$$

i.e. $T(r) := -500 \cdot r_2 + 10 \cdot \ln(r) + 80.2212$ ((b) ***...equation for temperature distribution.***)

Maximum temperature

Put $r = 0.1$ m in Eq. b:

$T(0.1) = 52.195$ *...same as obtained earlier...**verified.***

Temperature at inner surface, T_i:

Put $r = 0.05$ m in Eq. b:

$T(0.005) = 49.014$ *...same as obtained earliler ...**verified.***

Example 5.14. A thin, hollow tube with 4 mm inner diameter and 6 mm outer diameter, carries a current of 1000 amperes. Water at 30°C is circulated inside the tube for cooling the tube. Taking heat transfer coefficient on the water side as 35,000 W/(m^{2}C) and k for the material as 18 W/(mC), estimate the surface temperature of the tube if its outer surface is insulated. Electrical resistance of the tube is 0.0065 ohms per metre length.

Solution. See Figure Example 5.14.

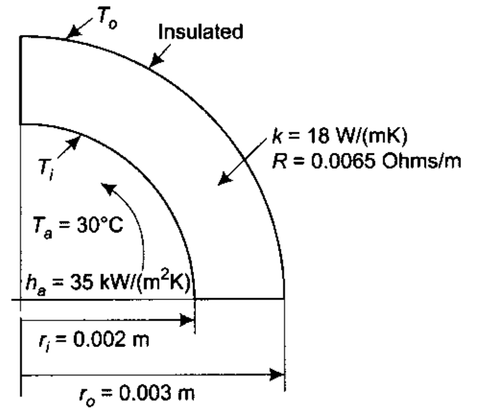

FIGURE Example 5.14 Hollow conductor with heat generation, losing heat on inside surface by convection, insulated on outside

Data:

$r_i := 0.002$ m $r_o := 0.003$ m $k := 18$ W/mK

$I := 1000$ A $R := 0.0065$ Ohms/m length

$T_a := 30°C$ $h_a := 35000$ W/(m^{2}K) $L := 1$ m

Problem is to find temperature on outer surface, T_o.

First, find q_g, heat generation rate per unit volume:

$$q_g = \frac{Q_g}{\text{Volume}}$$

where Q_g is total heat generated in the conductor volume.

i.e. $q_g = \dfrac{I^2 \cdot R}{\pi \cdot (r_o^2 - r_i^2) \cdot L}$ W/m^3 *(define q_g)*

i.e. $q_g = 4.18 \times 10^8$ W/m^3 *(heat generation rate per unit volume)*

To find outer surface temperature T_o:

This is the case of a hollow cylinder with heat generation, insulated on the outside. Therefore, Eq. 5.32 is applicable:

$$T_o - T_i = \frac{q_g \cdot r_o^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r_o}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - 1\right] \quad ...(5.32)$$

In the above equation RHS can be calculated, since all quantities on RHS are known.

However, T_i is not known yet; it is calculated by making an energy balance on the inner surface, remembering that all the heat generated in the shell flows only to the inner surface, since the outer surface is insulated:

i.e. heat generated in the shell = heat transferred to water from inner surface by convection

i.e. $$I^2 \cdot R = h_a \cdot (2 \cdot \pi \cdot r_i \cdot L) \cdot (T_i - T_a)$$

i.e. $$T_i := \frac{I^2 \cdot R}{h_a \cdot (2 \cdot \pi \cdot r_1 \cdot L)} + T_a °C$$ *(define inner surface temperature T_i)*

i.e. $T_i = 44.779°C$ ***(temperature of inner surface)***

Now, from Eq. 5.32:

$$T_o = T_i + \frac{q_g \cdot r_o^2}{4 \cdot k} \cdot \left[2 \cdot \ln\left(\frac{r_o}{r_i}\right) + \left(\frac{r_i}{r_o}\right)^2 - 1\right] °C$$ *(define T_o, temperature of outer surface)*

$T_o = 57.988°C$ ***(temperature of outer surface.)***

Exercise: Solve this problem, from fundamentals, i.e. starting from Eq. 5.18.

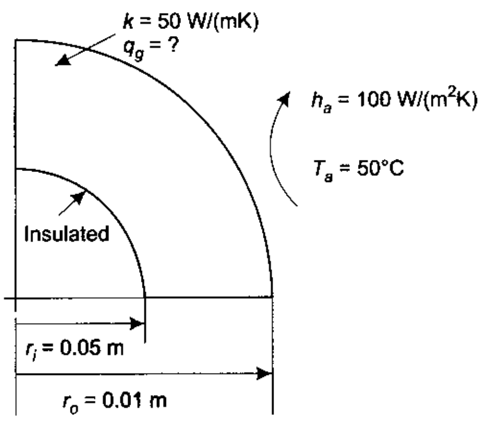

FIGURE Example 5.15 Hollow cylinder with heat generation, losing heat on outer surface by convection, inner surface insulated

Example 5.15. A nuclear fuel element is in the form of a hollow cylinder insulated at the inner surface. Its inner and outer radii are 5 cm and 10 cm, respectively. The outer surface gives heat to a fluid at 50°C where the unit surface conductance is 100 W/(m^{2}K). k of the material is 50 W/(mK). Find the rate of heat generation so that maximum temperature in the system will not exceed 200°C.

Solution. See Fig. Example 5.15.

Data:

$r_i := 0.05$ m $r_o := 0.1$ m $k := 50$ W/mK

$T_a := 50°C$ $h_a := 100$ W/(m^{2}K)

$L := 1$ m $T_i := 200°C$

To find rate of heat generation, q_g:

This is the case of a hollow cylinder with heat generation, insulated on the inside, losing heat on the outer surface to a fluid flowing at temperature T_a, with heat transfer coefficient h_a. Therefore, Eq. 5.29 is applicable:

$$T(r) = T_a + \frac{q_g \cdot (r_o^2 - r_i^2)}{2 \cdot h_a \cdot r_o} + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r} \right) - \left(\frac{r}{r_i} \right)^2 \right] \quad ...(5.29)$$

Here, T_i is known to be 200°C, since maximum temperature occurs on insulated inner surface, at r = 0.05 m. So, in Eq. 5.29 replace r by r_i and $T(r)$ by T_i; then, the only unknown, q_g can be calculated:

i.e.

$$T_i = T_a + \frac{q_g \cdot (r_o^2 - r_i^2)}{2 \cdot h_a \cdot r_o} + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i} \right) - \left(\frac{r_i}{r_i} \right)^2 \right]$$

Therefore,

$$q_g := \frac{T_i - T_a}{\frac{(r_o^2 - r_i^2)}{2 \cdot h_a \cdot r_o} + \frac{r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i} \right) - 1 \right]} \quad (\textit{define } q_g)$$

$q_g = 3.79582 \times 10^5$ W/m^3 = **379.582 kW/m^3** ***(heat generation rate/unit volume.)***

Exercise: Solve this problem, from fundamentals, i.e. starting from Eq. 5.18.

5.4 Sphere with Uniform Internal Heat Generation

Spherical geometry is popular for many applications, such as reactors for chemical processes, storage of radio-active wastes, experimental nuclear fuel elements, etc. We shall consider heat transfer in a solid sphere with different types of boundary conditions.

5.4.1 Solid Sphere with Internal Heat Generation

Consider a solid sphere of radius, R. There is uniform heat generation within its volume at a rate of q_g (W/m^3). Let the thermal conductivity, k be constant. See Fig. 5.12.

We would like to analyse this system for temperature distribution and maximum temperature attained.

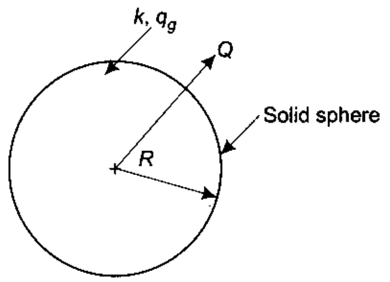

FIGURE 5.12(a) Spherical system with heat generation

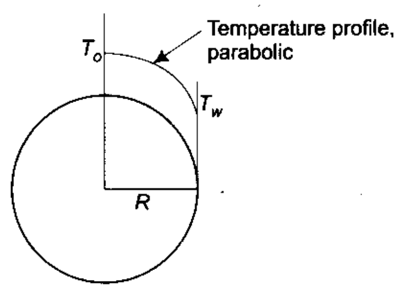

FIGURE 5.12(b) Variation of temperature along the radius

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the r direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Uniform internal heat generation rate, q_g (W/m^3).

With the above stipulations, the general differential equation in spherical coordinates (see Eq. 3.21) reduces to Eq. 3.24, i.e.

$$\frac{d^2T}{dr^2} + \frac{2}{r} \cdot \frac{dT}{dr} + \frac{q_g}{k} = 0 \quad ...(a)$$

We have to solve Eq. a to get the temperature profile; then, by applying Fourier's law, we can get the heat flux at any point.

Multiplying Eq. a by r^2: $$r^2 \cdot \frac{d^2T}{dr^2} + 2 \cdot r \cdot \frac{dT}{dr} + \frac{q_g \cdot r^2}{k} = 0$$

i.e. $$\frac{d}{dr}\left(r^2 \cdot \frac{dT}{dr}\right) = \frac{-q_g \cdot r^2}{k}$$

Integrating: $$r^2 \frac{dT}{dr} = \frac{-q_g \cdot r^3}{3 \cdot k} + C_1$$

i.e. $$\frac{dT}{dr} = \frac{-q_g \cdot r}{3 \cdot k} + \frac{C_1}{r^2} \quad \text{...(b)}$$

Integrating again: $$T(r) = \frac{-q_g \cdot r^2}{6 \cdot k} - \frac{C_1}{r} + C_2 \quad \text{...(5.37)}$$

Eq. 5.37 is the general relation for temperature distribution along the radius, for a spherical system, with uniform heat generation. C_1 and C_2, the constants of integration are obtained by applying the boundary conditions.

In the present case, B.C.'s are:

B.C. (i): at $r = 0$, $dT/dr = 0$, i.e. at the centre of the sphere, temperature is finite and maximum (i.e. $T_o = T_{max}$) because of symmetry (heat flows from inside to outside radially, in all directions).

B.C. (ii): at $r = R$, i.e. at the surface, $T = T_w$

From B.C. (i) and Eq. b, we get: $C_1 = 0$

From B.C. (ii) and Eq. 5.37, we get:

$$T_w = \frac{-q_g \cdot R^2}{6 \cdot k} + C_2$$

i.e. $$C_2 = T_w + \frac{q_g \cdot R^2}{6 \cdot k}$$

Substituting C_1 and C_2 in Eq. 5.37:

$$T(r) = \frac{-q_g \cdot r^2}{6 \cdot k} + T_w + \frac{q_g \cdot R^2}{6 \cdot k}$$

i.e. $$T(r) = T_w + \frac{q_g}{6 \cdot k} \cdot (R^2 - r^2) \quad \text{...(5.38)}$$

Eq. 5.38 is the relation for temperature distribution for a solid sphere, in terms of the surface temperature, T_w. Note that this is a parabolic temperature profile, as shown in Fig. 5.12(b).

Maximum temperature:

Maximum temperature occurs at the centre, because of symmetry considerations (i.e. heat flows from the centre radially outward in all directions; therefore, temperature at the centre must be the maximum).

Therefore, putting $r = 0$ in Eq. 5.38:

$$T_{max} = T_w + \frac{q_g \cdot R^2}{6 \cdot k} \quad \text{...(5.39)}$$

From Eqs. 5.38 and 5.39,

$$\frac{T(r) - T_w}{T_{max} - T_w} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{...(5.40)}$$

Eq. 5.40 is the non-dimensional temperature distribution for the solid sphere with heat generation.

Heat flow at the surface:

Of course, in steady state, heat transfer rate at the surface must be equal to the heat generation rate in the sphere,

$$Q_g = \left(\frac{4}{3}\right)\pi R^3 q_g \qquad ...(c)$$

Heat transfer by conduction at the outer surface of sphere is given by Fourier's law:

i.e. $$Q_g = -kA(dT/dr)|_{\text{at } r=R}$$

i.e. $$Q_g = -k\cdot 4\cdot\pi\cdot R^2\cdot\left(\frac{-q_g\cdot R}{3\cdot k}\right) \qquad \textit{(using Eq. 5.38 for } T(r))$$

i.e. $$Q_g = \frac{4}{3}\cdot\pi\cdot R^3\cdot q_g \qquad ...(5.41)$$

Eq. 5.41 and Eq. c are the same, as expected.

Convection boundary condition:

When heat is carried away at the outer surface by a fluid at a temperature T_a flowing on the surface with a convective heat transfer coefficient, h_a. (e.g. hot spherical ball cooled by ambient air), then, mostly, it is the fluid temperature that is known and not the surface temperature, T_w, of the sphere. In such cases, we relate the wall temperature and fluid temperature by an energy balance at the surface, i.e. heat generated and conducted from within the body to the surface is equal to the heat convected away by the fluid at the surface.

i.e. $$\frac{4}{3}\cdot\pi\cdot R^3\cdot q_g = h_a\cdot(4\cdot\pi\cdot R^2)\cdot(T_w - T_a)$$

i.e. $$T_w = T_a + \frac{q_g\cdot R}{3\cdot h_a} \qquad ...(d)$$

Substituting in Eq. 5.38:

$$T(r) = T_a + \frac{q_g\cdot R}{3\cdot h_a} + \frac{q_g}{6\cdot k}\cdot(R^2 - r^2) \qquad ...(5.42)$$

Again, for maximum temperature put $r = 0$ in Eq. 5.42:

$$T_{max} = T_a + \frac{q_g\cdot R}{3\cdot h_a} + \frac{q_g\cdot R^2}{6\cdot k} \qquad ...(5.43)$$

Eq. 5.43 gives the centre temperature of the sphere with heat generation, in terms of the fluid temperature, when the heat generated is carried away at the surface by a fluid.

5.4.2 Alternative Analysis

In the alternative method, which is simpler, instead of starting with the general differential equation, we derive the above equations from physical considerations. See Fig. 5.13.

Let us write an energy balance with an understanding that at any radius r, the amount of heat generated in the volume within $r = 0$ and $r = r$, must move outward by conduction.

$$\frac{4}{3}\cdot\pi\cdot r^3\cdot q_g = -k\cdot(4\cdot\pi\cdot r^2)\cdot\frac{dT}{dr} \qquad ...(a)$$

i.e. $$dT = \frac{-q_g}{3\cdot k}\cdot r\cdot dr$$

Integrating: $$\int dT = \frac{-q_g}{3\cdot k}\int r\,dr$$

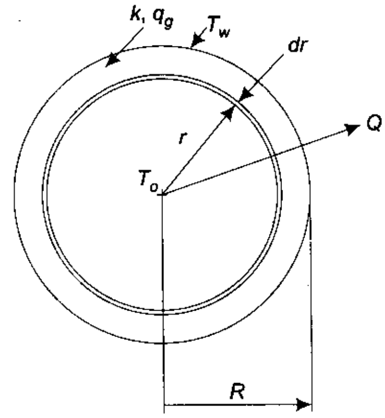

FIGURE 5.13 Solid sphere with heat generation

i.e. $$T(r) = \frac{-q_g \cdot r^2}{6 \cdot k} + C \qquad ...(b)$$

Eq. b gives the temperature distribution along the radius.

Get the constant of integration, C from the B.C.: at $r = R$, $T = T_w$

Then, from Eq. b:

i.e. $$C = T_w + \frac{q_g \cdot R^2}{6 \cdot k}$$

Substituting back in Eq. b:

$$T(r) = \frac{-q_g \cdot r^2}{6 \cdot k} + T_w + \frac{q_g \cdot R^2}{6 \cdot k}$$

i.e. $$T(r) = T_w + \frac{q_g}{6 \cdot k} \cdot (R^2 - r^2) \qquad ...(c)$$

Eq. c gives the temperature distribution along the radius, in terms of the surface temperature of the cylinder. Note that Eq. c is the same as Eq. 5.38 derived earlier.

In many cases, temperature drop between the centre of the sphere (where maximum temperature occurs) and the surface is important. Then, from Eq. c, putting $r = 0$:

$$T_{max} = T_w + \frac{q_g \cdot R^2}{6 \cdot k} \qquad ...(d)$$

Eq. d is the same as Eq. 5.39.

And, from Eqs. c and d, we can write:

$$\frac{T(r) - T_w}{T_{max} - T_w} = 1 - \left(\frac{r}{R}\right)^2 \qquad ...(e)$$

Eq. e is the same as Eq. 5.40, and gives the non-dimensional temperature distribution in the sphere with heat generation. If heat generated in the sphere is carried away by convection, by a fluid flowing on the surface of the sphere, the wall temperature and fluid temperature are related by an energy balance at the surface, as done earlier.

5.4.3 Analysis with Variable Thermal Conductivity

In the above analysis, thermal conductivity of the material was assumed to be constant. Now, let us make an analysis when the thermal conductivity varies linearly with temperature as:

$$k(T) = k_o(1 + \beta T),$$

where, k_o and β are constants.

Again, considering Fig. 5.13, we have from heat balance (see Eq. a above):

$$\frac{4}{3} \cdot \pi \cdot r^3 \cdot q_g = -k(T) \cdot (4 \cdot \pi \cdot r^2) \cdot \frac{dT}{dr} \qquad ...(a)$$

i.e. $$k(T) \cdot dT = \frac{-q_g}{3} \cdot r \cdot dr$$

Substituting for $k(T)$ and integrating:

$$\int k_o \cdot (1 + \beta \cdot T) dT = \frac{-q_g}{3} \int r dr$$

i.e. $$T + \frac{\beta \cdot T^2}{2} = \frac{-q_g \cdot r^2}{6 \cdot k_o} + C \qquad ...(f)$$

C is determined from the B.C.: at $r = 0$, $T = T_o$

We get:

$$C = T_o + \frac{\beta \cdot T_o^2}{2}$$

Substituting C in Eq. f:

$$\frac{\beta \cdot T^2}{2} + T + \frac{q_g \cdot r^2}{6 \cdot k_o} - T_o - \frac{\beta \cdot T_o^2}{2} = 0 \qquad \text{...(g)}$$

Eq. g is a quadratic in T. Its positive root is given by:

$$T(r) = \frac{-1 + \sqrt{1 - 4 \cdot \frac{\beta}{2} \cdot \left(\frac{q_g \cdot r^2}{6 \cdot k_o} - T_o - \frac{\beta \cdot T_o^2}{2} \right)}}{2 \cdot \frac{\beta}{2}}$$

i.e.

$$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta^2} + T_o^2 + \frac{2 \cdot T_o}{\beta} \right) - \frac{q_g \cdot r^2}{3 \cdot \beta \cdot k_o}}$$

i.e.

$$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_o \right)^2 - \frac{q_g \cdot r^2}{3 \cdot \beta \cdot k_o}} \qquad \text{...(5.44)}$$

Eq. 5.44 gives temperature distribution in a solid sphere with internal heat generation and linearly varying k. Compare this equation with that obtained for a slab, with temperature at either side being the same, i.e. Eq. 5.10 and that for a solid cylinder, i.e. Eq. 5.25.

Eq. 5.44 gives $T(r)$ in terms of T_o (i.e. T_{max} at $r = 0$).

If we need $T(r)$ in terms of T_w, then in Eq. f, use the B.C: at $r = R$, $T = T_w$

Then we get:

$$C = T_w + \frac{\beta \cdot T_w^2}{2} + \frac{q_g \cdot R^2}{6 \cdot k_o}$$

Substitute this in Eq. f and get a quadratic in T.

Solving, we get, for temperature distribution:

$$T(r) = \frac{-1}{\beta} + \sqrt{\left(\frac{1}{\beta} + T_w \right)^2 + \frac{q_g \cdot (R^2 - r^2)}{3 \cdot \beta \cdot k_o}} \qquad \text{...(5.45)}$$

Example 5.16. A solid sphere of radius, R = 10 mm and k = 18 W/(mC) has a uniform heat generation rate of 2×10^6 W/m^3. Heat is conducted away at its outer surface to ambient air at 20°C by convection, with a heat transfer coefficient of 2000 W/(m^{2}C).

(i) Determine the steady state temperature at the centre and outer surface of the sphere.

(ii) Draw the temperature profile along the radius.

Solution. See Figure Example 5.16.

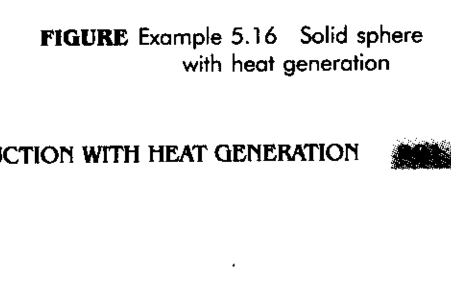

FIGURE Example 5.16 Solid sphere with heat generation

Data:

$R := 0.01$ m $\quad h_a := 2000$ W/(m^{2}K) $\quad k := 18$ W/mK)

$T_a := 20$°C $\quad q_g := 2 \times 10^6$ W/m^3

To calculate T_a and T_w:

From Eq. 5.39, we have

$$T_{max} = T_w + \frac{q_g \cdot R^2}{6 \cdot k} \qquad \text{...(5.39)}$$

And, from heat balance on the surface of the sphere,

$$\frac{4}{3}\cdot\pi\cdot R^3\cdot q_g = h_a\cdot(4\cdot\pi\cdot R^2)\cdot(T_w - T_a)$$

i.e. $$T_w := T_a + \frac{q_g\cdot R}{3\cdot h_a}$$

i.e. $$T_w = 23.\ 333°C$$ *(surface temperature of sphere.)*

Therefore, $$T_{max} := T_a + \frac{q_g\cdot R}{3\cdot h_a} + \frac{q_g\cdot R^2}{6\cdot k}$$

i.e. $$T_{max} = 25.185°C$$ *(centre temperature of sphere.)*

To sketch the temperature profile:

Temperature distribution is given by Eq. 5.42, i.e.

$$T(r) := T_a + \frac{q_g\cdot R}{3\cdot h_a} + \frac{q_g}{6\cdot k}\cdot(R^2 - r^2) \quad ...(5.42)$$

To sketch the temperature profile in the sphere, define a range variable r, varying from 0 to 0.01 m, with an increment of 0.0005 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears: See Fig. Ex. 5.16(b)

$$r := 0, 0.0005, ..., 0.01$$ *(define a range variable r..starting value = 0, next value = 0.0005 m and last value = 0.01 m)*

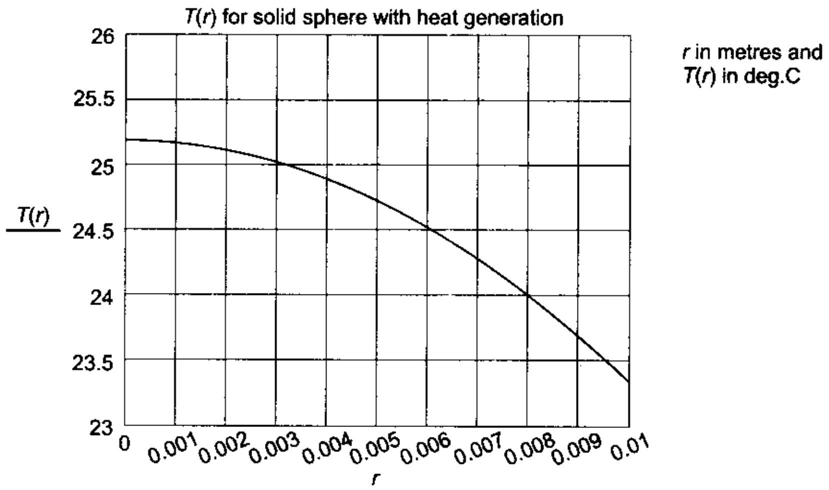

FIGURE Example 5.16(b)

It may be verified from the graph that temperature of the centre and outside surface of the sphere are 25.19°C and 23.33°C, respectively.

Example 5.17. In a sphere of radius R, heat generation rate varies with the radius as: $q_g = q_o\,[1-(r/R)^2]$. If the thermal conductivity k, is constant, derive an expression for the variation of temperature with radius.

Solution. This is a case of solid sphere with variable rate of heat generation.

See Figure Example 5.17.

The method is, as usual, to start with the governing equation for the assumtions of the problem, namely, one-dimensional, steady state conduction with heat generation, with constant k, in spherical coordinates:

i.e. $$\frac{d^2T}{dr^2} + \frac{2}{r}\cdot\frac{dT}{dr} + \frac{q_g}{k} = 0 \quad ...(a)$$

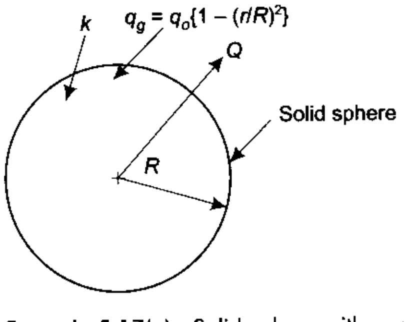

FIGURE Example 5.17(a) Solid sphere with variable heat generation

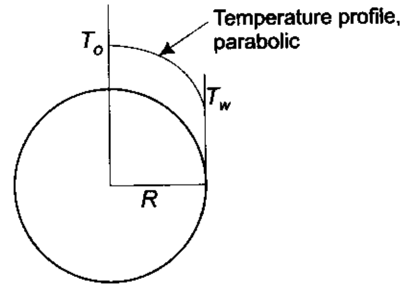

FIGURE Example 5.17(b) Variation of temperature along the radius

Multiplying Eq. a by r^2: $r^2 \cdot \dfrac{d^2T}{dr^2} + 2 \cdot r \cdot \dfrac{dT}{dr} + \dfrac{q_g \cdot r^2}{k} = 0$

i.e.
$$\frac{d}{dr}\left(r^2 \cdot \frac{dT}{dr}\right) + \frac{q_g \cdot r^2}{k} = 0$$

Substituting for q_g: $\dfrac{d}{dr}\left(r^2 \cdot \dfrac{dT}{dr}\right) + \dfrac{q_o \cdot \left[1 - \left(\dfrac{r}{R}\right)^2\right] \cdot r^2}{k} = 0$

Integrating:
$$r^2 \cdot \frac{dT}{dr} + \frac{q_o \cdot r^3}{3 \cdot k} - \frac{q_o \cdot r^5}{5 \cdot k \cdot R^2} = C_1$$

i.e.
$$\frac{dT}{dr} + \frac{q_o \cdot r}{3 \cdot k} - \frac{q_o \cdot r^3}{5 \cdot k \cdot R^2} = \frac{C_1}{r^2} \qquad \text{...(b)}$$

Integrating again:
$$T(r) = \frac{-q_o \cdot r^2}{6 \cdot k} + \frac{q_o \cdot r^4}{20 \cdot k \cdot R^2} - \frac{C_1}{r} + C_2 \qquad \text{...(c)}$$

Eq. c gives the temperature distribution. Obtain C_1 and C_2 by applying the B.C.'s:
B.C. (i): at $r = 0$, $dT/dr = 0$, since temperature is maximum at the centre due to symmetry.
B.C. (ii): at $r = R$, $T = T_w$
From B.C. (i) and Eq. b, we get $C_1 = 0$
From B.C. (ii) and Eq. c

$$C_2 = T_w + \frac{q_o \cdot R^2}{6 \cdot k} - \frac{q_o \cdot R^2}{20 \cdot k}$$

Substituting for C_1 and C_2 in Eq. c:

$$T(r) = T_w - \frac{q_o \cdot r^2}{6 \cdot k} + \frac{q_o \cdot r^4}{20 \cdot k \cdot R^2} + \frac{q_o \cdot R^2}{6 \cdot k} - \frac{q_o \cdot R^2}{20 \cdot k}$$

i.e.
$$T(r) = T_w + \frac{q_o}{6 \cdot k} \cdot (R^2 - r^2) - \frac{q_o}{20 \cdot k \cdot R^2} \cdot (R^4 - r^4) \qquad \text{...(d)}$$

Eq. d gives the desired temperature distribution in the sphere.
When $r = 0$, $T = T_o = T_{max}$. Then, from Eq. d:

$$T_o - T_w = \frac{q_o}{6 \cdot k} \cdot (R^2) - \frac{q_o \cdot R^2}{20 \cdot k} = \frac{7}{60} \cdot \frac{q_o \cdot R^2}{k} \qquad \text{...(e)}$$

Eq. e gives the maximum temperature difference in the sphere with heat generation varying with position as: $q_g = q_o \cdot [1 - (r/R)^2]$.

Example 5.18. In Example 5.17, if $q_o = 10^6$ W/m^3, $R = 0.04$ m, $k = 12$ W/(mC), and if the centre temperature is 200°C, determine the surface temperature. Also, find the heat flow rate at the surface. Draw the temperature profile.

Solution.

Data:

$$R := 0.04 \text{ m} \qquad q_g := q_o \cdot \left[1 - \left(\frac{r}{R}\right)^2\right] \qquad q_o := 10^6 \text{ W/m}^3 \qquad k := 12 \text{ W/(mC)} \qquad T_o := 200°\text{C}$$

To find the surface temperature:

Apply the Eq. d developed in the previous Example:

i.e.
$$T(r) = T_w + \frac{q_o}{6 \cdot k} \cdot (R^2 - r^2) - \frac{q_o}{20 \cdot k \cdot R^2} \cdot (R^4 - r^4) \qquad ...(d)$$

Temperature is maximum when $r = 0$:

i.e.
$$T_o = T_w + \frac{q_o}{6 \cdot k} \cdot R^2 - \frac{q_o}{20 \cdot k} \cdot R^2$$

Therefore,
$$T_w := T_o - \frac{q_o}{6 \cdot k} \cdot R^2 + \frac{q_o}{20 \cdot k} R^2$$

i.e. $T_w = 184.444°\text{C}$ ***(temperature in the surface of sphere)***

Heat flow rate at the surface, Q:

Apply Fourier's law at the surface, since, now, we have equation for temperature distribution:

i.e.
$$Q = -k \cdot 4 \cdot \pi \cdot R^2 \cdot \left.\frac{dT(r)}{dr}\right|_{r=R}$$

Now, we have:

$$T(r) := \left[T_w + \frac{q_o}{6 \cdot k} \cdot (R^2 - r^2) - \frac{q_o}{20 \cdot k \cdot R^2} \cdot (R^4 - r^4)\right]$$

In Mathcad, we do not have to actually differentiate and expand the expression.

But, define $T'(r) = dT(r)/dr$ and find out $T'(r)$ at $r = R$:

$$T'(r) := \frac{d}{dr} T(r)$$ *(define the first derivative of $T(r)$ w.r.t. T)*

Therefore, $T'(R) = -444.444$ *(value of $T'(r)$ at $r = R = 0.04$ m)*

Therefore, $Q := -k \cdot 4 \cdot \pi \cdot R^2 \cdot T'(R)$, W *(define heat transfer rate at the surface)*

i.e. $Q = 107.233$ W ***(heat transfer rate at the surface.)***

To sketch the temperature profile in the sphere, define a range variable r, varying from 0 to 0.04 m, with an increment of 0.001 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears.

$r := 0, 0.001, ..., 0.04$ *(define a range variable r..starting value = 0, next value = 0.001 m and last value = 0.04 m)*

It may be verified from the graph that temperature at the centre and on the outside surface of the sphere are 200°C and 184.44°C, respectively.

5.5 Applications

In this chapter, so far, we studied the steady state, one-dimensional heat transfer, with internal heat generation in simple geometries such as slabs, cylinders and spheres. Now, we shall analyse some practical examples based on these geometries.

5.5.1 Dielectric Heating

Dielectric heating is a very popular, industrial method of heating adopted to heat insulating materials such as wool, rubber, plastics and textiles. Here, a high frequency, high voltage alternating current is applied to the plates of a condenser; the insulating material to be heated is placed between the plates. Heat is generated within the volume at an uniform rate.

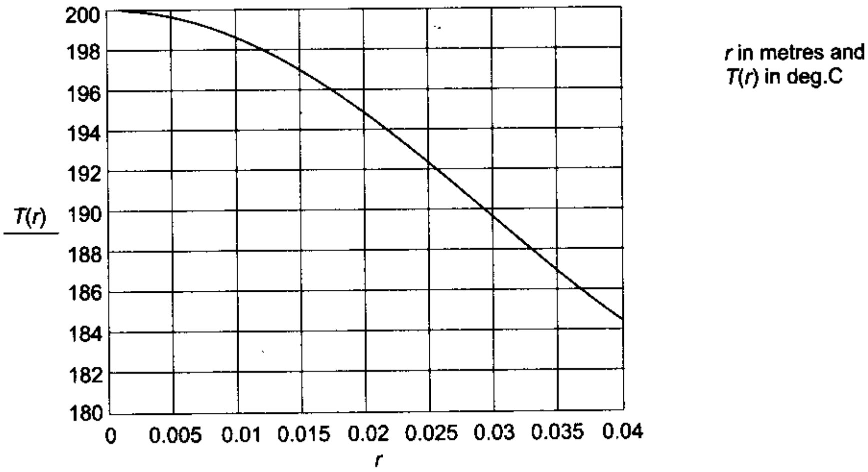

FIGURE Example 5.17(b) $T(r)$ vs. r for sphere-variable heat generation

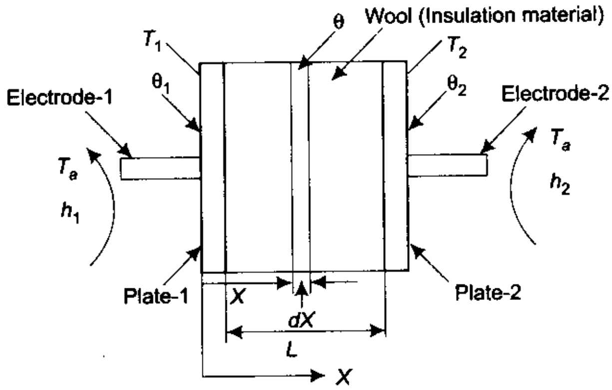

FIGURE 5.14 Dieletric heating

Refer to Fig. 5.14. The insulation material of thickness L is placed between the two electrodes 1 and 2 and high frequency, high voltage, alternating current is applied. Plates 1 and 2 will also get heated up while the insulation material is uniformly heated up at a rate of q_g (W/m^3). Both the plates lose heat to the ambient air at temperature T_a, with heat transfer coefficients of h_1 and h_2, respectively. Let the plate temperatures be T_1 and T_2, as shown.

It is clear that this situation is similar to a plane wall with uniform heat generation and we shall use the general differential equation for conduction in Cartesian coordinates, with the following assumptions:

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the x direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Uniform internal heat generation rate, q_g (W/m^3).

Consider any section within the volume at a distance x from the origin. Let the temperature at this section be T.

Now, with the above assumptions, the controlling differential equation is:

$$\frac{d^2T}{dx^2}+\frac{q_g}{k}=0 \quad ...(a)$$

Let us define excess temperature, $\theta = T - T_a$, where T_a is constant ambient temperature.

Then, $\theta_1 = T_1 - T_a$, and

$\theta_2 = T_2 - T_a$

Also,

$$\frac{d\theta}{dx}=\frac{dT}{dx}, \text{ and}$$

$$\frac{d^2\theta}{dx^2}=\frac{d^2T}{dx^2}$$

Then, Eq. a may be written as:

$$\frac{d^2\theta}{dx^2}+\frac{q_g}{k}=0 \quad ...(b)$$

Integrating, $\dfrac{d\theta}{dx}+\dfrac{q_g\cdot x}{k}=C_1$

Integrating again, $$\theta+\frac{q_g\cdot x^2}{2\cdot k}=C_1\cdot x+C_2 \quad ...(c)$$

Eq. c gives the temperature distribution in the medium.

Integration constants, C_1 and C_2 are obtained from the B.C's.

B.C.(i): at $x = 0$, heat conducted must be equal to the heat removed by convection from Plate 1 to the ambient.

i.e. $$+kA\left(\frac{d\theta}{dx}\right)\bigg|_{x=0}=h_1A(T_1-T_a)=h_1A\,\theta_1$$

(Note that positive sign is used on the LHS of Fourier's equation above, since the heat flow on the left plate is from right to left, i.e. in the negative x-direction).

i.e. $$k\cdot A\cdot C_1=h_1\cdot A\cdot\theta_1 \quad ...since\ \frac{d\theta}{dx}\bigg|_{x=0}=C_1$$

or, $$C_1=\frac{h_1\cdot\theta_1}{k} \quad ...(d)$$

B.C.(ii): at $x = 0$, $\theta = \theta_1$

Therefore, from Eq. c:

$$C_2=\theta_1 \quad ...(e)$$

Substituting C_1 and C_2 in Eq. c:

$$\theta(x)=\frac{-q_g\cdot x^2}{2\cdot k}+\frac{h_1\cdot\theta_1\cdot x}{k}+\theta_1 \quad ...(5.46)$$

Eq. 5.46 gives the temperature distribution in the medium, in terms of θ_1.

θ_2 for the plate on the right is obtained by putting $x = L$ and $\theta = \theta_2$ in Eq. 5.46.

i.e. $$\theta_2=\frac{-q_g\cdot L^2}{2\cdot k}+\frac{h_1\cdot\theta_1\cdot L}{k}+\theta_1 \quad ...(f)$$

It is obvious that in steady state, total heat generated within the medium must be equal to the sum of heat convected away at the left and right plates:

i.e. $$q_g \cdot L \cdot A = h_1 \cdot A \cdot \theta_1 + h_2 \cdot A \cdot \theta_2$$

or, $$q_g \cdot L = h_1 \cdot \theta_1 + h_2 \cdot \theta_2 \quad \text{...(g)}$$

Electrode temperatures T_1 and T_2 are obtained by solving Eqs. f and g simultaneously.

5.5.2 Heat Transfer through a Piston Crown

Cylinder and piston arrangement is shown in Fig. 5.15.

Piston crown is subjected to a uniform heat flux due to convection and radiation from the hot gases and cylinder walls. Let this heat flux be q_g (W/m^2). Let the outside radius of the piston crown be R and its thickness, b. Let T_o be the temperature of outer surface of the crown, and k, the thermal conductivity of the crown material.

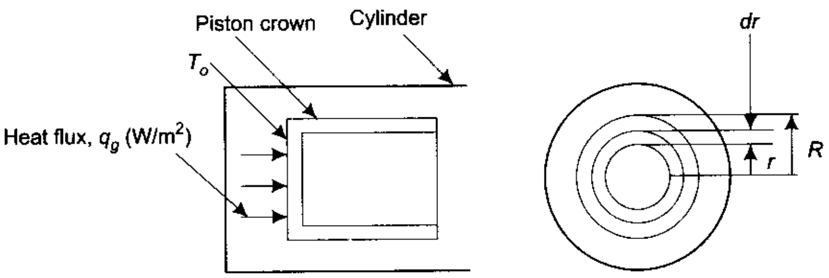

FIGURE 5.15 Heat transfer through piston crown

To derive the differential equation governing the temperature distribution in the crown, let us follow the usual procedure of writing an energy balance on an infinetisimal control volume:

Consider an elemental volume at radius r and of width dr as shown.

Heat conducted into the element at radius, r:

$$Q_r = -k \cdot (2 \cdot \pi \cdot r \cdot b) \cdot \frac{dT}{dr}$$

(Remember that area in Fourier's equation is the area normal to the direction of heat flow = $(2\pi r b)$).

Heat given by gases to the element:

$$Q_g = q_g \cdot (2 \cdot \pi \cdot r \cdot dr)$$

Heat conducted out of the element at radius $(r + dr) = Q|_{r+dr} =$

$$Q_r + \frac{d}{dr} \cdot (Q_r) \cdot dr$$

Then, in steady state, writing an energy balance:

$$Q_r + Q_g = Q|_{r+dr}$$

$$= Q_r + \left(\frac{dQ_r}{dr}\right).dr$$

Therefore, $$Q_g = \frac{d}{dr} \cdot (Q_r) \cdot dr$$

i.e. $$q_g \cdot 2 \cdot \pi \cdot r \cdot dr = \frac{d}{dr} \cdot \left(-k \cdot 2 \cdot \pi \cdot r \cdot b \cdot \frac{dT}{dr}\right) \cdot dr$$

i.e. $$\frac{d}{dr}\left(r \cdot \frac{dT}{dr}\right) + \frac{q_g \cdot r}{k \cdot b} = 0 \quad \text{...(a)}$$

Integrating, $$\frac{dT}{dr} + \frac{q_g \cdot r}{2 \cdot k \cdot b} = \frac{C_1}{r} \quad \text{...(b)}$$

Integrating again, $$T(r) + \frac{q_g \cdot r^2}{4 \cdot k \cdot b} = C_1 \cdot \ln(r) + C_2 \quad \text{...(c)}$$

Eq. c gives the temperature distribution along the radius for the piston crown. C_1 and C_2 are determined from the B.C.'s:

B.C.(i): at $r = 0$, $\frac{dT}{dr} = 0$ since temperature is a maximum at the centre by symmetry (i.e. heat flows from centre to periphery radially).

B.C.(ii): at $r = R$, $T = T_o$

From B.C.(i) and Eq. b: $C_1 = 0$

From B.C.(ii) and Eq. c:

$$T_o + \frac{q_g \cdot R^2}{4 \cdot k \cdot b} = C_2$$

Substituting C_1 and C_2 in Eq. c

$$T(r) + \frac{q_g \cdot r^2}{4 \cdot k \cdot b} = T_o + \frac{q_g \cdot R^2}{4 \cdot k \cdot b}$$

i.e. $$T(r) = T_o + \frac{q_g}{4 \cdot k \cdot b} \cdot (R^2 - r^2) \quad \text{...(5.47)}$$

Eq. 5.47 gives the temperature distribution along the radius for the piston crown.

Note that the temperature distribution is parabolic.

Maximum temperature:

Maximum temperature occurs at the centre, i.e. at $r = 0$.

Putting $r = 0$ in Eq. 5.47, we get:

$$T_{max} = T_o + \frac{q_g \cdot R^2}{4 \cdot k \cdot b} \quad \text{...(5.48)}$$

If Q is the total heat given by gases to the piston crown, then,

$$Q = \pi \cdot R^2 \cdot q_g$$

i.e. $$q_g = \frac{Q}{\pi \cdot R^2}$$

Therefore, $$T_{max} = T_0 + \frac{Q}{\pi \cdot R^2} \cdot \frac{R^2}{4 \cdot k \cdot b} \quad \text{...(5.49)}$$

And, thickness of piston crown:

$$b = \frac{Q}{4 \cdot \pi \cdot k \cdot (T_{max} - T_o)} \quad \text{...(5.50)}$$

Eq. 5.50 is important, since it gives the thickness required for the piston crown in terms of Q, T_{max} and T_o.

5.5.3 Heat Transfer in Nuclear Fuel Rod (without cladding)

In a nuclear fuel rod, heat is generated by slowing down of neutrons in a fissionable material; however, this heat generated is not uniform throughout the material, but, varies with position according to the following relation:

$$q_g = q_o \cdot \left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{...(a)}$$

where, q_o = heat generation rate per unit volume at the centre (i.e. at $r = 0$), and

R = outer radius of the solid fuel rod.

We would like to get an expression for the temperature distribution in the fuel rod, maximum temperature in the rod and, of course, the total heat transferred. See Fig. 5.16.

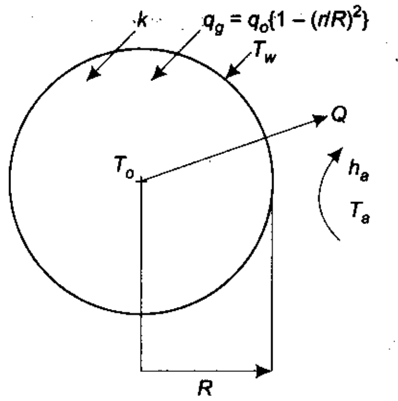

FIGURE 5.16 Cylindrical fuel rod with heat generation varying with position

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the r direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Internal heat generation at a varying rate: $q_g = q_o \cdot \{1 - (r/R)^2\}$, (W/m³).

For these assumptions, the controlling differential equation in cylindrical coordinates becomes:

$$\frac{d^2T}{dr^2} + \frac{1}{r} \cdot \frac{dT}{dr} + \frac{q_g}{k} = 0 \qquad ...(b)$$

Multiplying by r: $r \cdot \frac{d^2T}{dr^2} + \frac{dT}{dr} + \frac{q_g \cdot r}{k} = 0$

i.e. $$\frac{d}{dr}\left(r \cdot \frac{dT}{dr}\right) + \frac{q_g \cdot r}{k} = 0$$

i.e. $$\frac{d}{dr}\left(r \cdot \frac{dT}{dr}\right) + \frac{q_o \cdot r}{k} \cdot \left[1 - \left(\frac{r}{R}\right)^2\right] = 0$$

Integrating: $$r \cdot \frac{dT}{dr} + \frac{q_o}{k} \cdot \left(\frac{r^2}{2} - \frac{r^4}{4 \cdot R^2}\right) = C_1 \qquad ...(c)$$

i.e. $$\frac{dT}{dr} + \frac{q_o}{k} \cdot \left(\frac{r}{2} - \frac{r^3}{4 \cdot R^2}\right) = \frac{C_1}{r}$$

Integrating again, $$T(r) + \frac{q_o}{k} \cdot \left(\frac{r^2}{4} - \frac{r^4}{16 \cdot R^2}\right) = C_1 \cdot \ln(r) + C_2 \qquad ...(d)$$

Eq. d gives temperature profile within the fuel rod. C_1 and C_2 are obtained from the B.C.'s:

B.C.(i) at $r = 0$, $\frac{dT}{dr} = 0$ since temperature is a maximum at the centre of the rod.

B.C.(ii) Also, at $r = 0$, $T = T_{max}$

Then, from Eq. c, $C_1 = 0$

And, from Eq. d: $C_2 = T_{max}$

Substituting C_1 and C_2 in Eq. d:

$$T(r) + \frac{q_o}{k} \cdot \left(\frac{r^2}{4} - \frac{r^4}{16 \cdot R^2}\right) = T_{max}$$

i.e. $$T(r) - T_{max} = \frac{-q_o}{k} \cdot \left(\frac{r^2}{4} - \frac{r^4}{16 \cdot R^2}\right) \qquad ...(5.51)$$

Eq. 5.51 gives the temperature distribution in terms of the centre temperature of the fuel rod.

Surface temperature T_w:

Surface temperature of the rod is obtained by replacing r by R and $T(r)$ by T_w in Eq. 5.51:

i.e. $$T_{max} - T_w = \frac{q_o}{k} \cdot \left(\frac{R^2}{4} - \frac{R^4}{16 \cdot R^2} \right)$$

i.e. $$T_{max} - T_w = \frac{3 \cdot q_o \cdot R^2}{16 \cdot k} \quad ...(5.52)$$

Eq. 5.52 is an important relation, since it gives the maximum temperature drop in the fuel rod. It is necessary to know this quantity to ensure that sufficient cooling is provided, so that the fuel rod does not get overheated or melt down.

Heat flow from the surface:

Knowing the temperature distribution, heat flow rate at any point is obtained by applying the Fourier's law:

At the surface, i.e. at $r = R$:

$$Q = -kA \left(\frac{dT}{dr} \right)_{r=R}$$

i.e. $$Q = k \cdot A \cdot \left[\frac{q_o}{k} \cdot \left(\frac{R}{2} - \frac{R^3}{4 \cdot R^2} \right) \right]$$

i.e. $$Q = \frac{q_o \cdot A \cdot R}{4} \quad ...(5.53)$$

Convection boundary conditions:

If the heat generated is carried away at the surface by a fluid at temperature T_a, flowing with a convective heat transfer coefficient of h_a, we write the energy balance in steady state, i.e.

· Heat generated in the rod = Heat carried away by convection at the surface.

i.e. $$\frac{q_o \cdot A \cdot R}{4} = h_a \cdot A \cdot (T_w - T_a)$$

i.e. $$T_w = T_a + \frac{q_o \cdot R}{4 \cdot h_a}$$

Substituting this value of T_w in Eq. 5.52, we get:

$$T_{max} - T_a = \frac{q_o \cdot R}{4 \cdot h_a} + \frac{3 \cdot q_o \cdot R^2}{16 \cdot k}$$

i.e. $$T_{max} - T_a = \frac{q_o \cdot R}{4} \cdot \left(\frac{1}{h_a} + \frac{3 \cdot R}{4 \cdot k} \right) \quad ...(5.54)$$

Eq. 5.54 gives the maximum temperature (i.e. at the centre) in the fuel rod, in terms of the fluid temperature.

Example 5.19. A cylindrical fuel rod is of 20 cm diameter and has $k = 40$ W/(mK). Surface temperature of the rod is 75°C. Heat generation rate in the rod is given by:
$q_g = q_o\{1 - (r/R)^2\}$, where $q_o = 5.25 \times 10^6$ W/m^3. Determine the temperature at the centre of the rod, and the heat transfer rate per metre length of rod. Also, draw the temperature profile.

Soution. See Figure Example 5.19.

Data:

$R := 0.1$ m $\quad T_w := 75°C \quad k := 40$ W/(mK) $\quad q_g = q_o \cdot \left[1 - \left(\frac{r}{R} \right)^2 \right]$ W/m^3

$q_o := 5.25 \times 10^6$ W/m^3 $\quad L := 1$ m

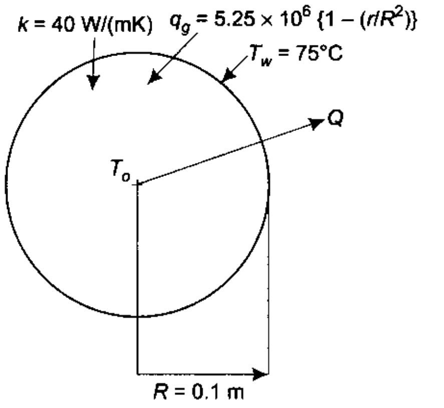

FIGURE Example 5.19 Cylindrical fuel rod with heat generation varying with position

Temperature at the centre of rod:
We use Eq. 5.52:

i.e. $$T_{max} - T_w = \frac{3 \cdot q_o \cdot R^2}{16 \cdot k} \quad \text{...(5.52)}$$

i.e. $$T_{max} := T_w + \frac{3 \cdot q_o \cdot R^2}{16 \cdot k}$$

i.e. T_{max} = 321.094°C (***temperature at the centre of the rod.***)

Heat transfer rate per metre length of rod:
We use Eq. 5.53:

i.e. $$Q = \frac{q_o \cdot A \cdot R}{4} \quad \text{...(5.53)}$$

Therefore, $$Q := \frac{q_o \cdot (2 \cdot \pi \cdot R \cdot L) \cdot R}{4} \ \text{W/m} \quad (\textit{define } Q)$$

i.e. $Q = 8.24668 \times 10^4$ W ... = 82.4668 KW/m (***heat transfer rate/m.***)

To draw temperature profile:
We use Eq. 5.51:

i.e. $$T(r) - T_{max} = \frac{-q_o}{k} \cdot \left(\frac{r^2}{4} - \frac{r^4}{16 \cdot R^2} \right) \quad \text{...(5.51)}$$

i.e. $$T(r) := T_{max} - \frac{q_o}{k} \cdot \left(\frac{r^2}{4} - \frac{r^4}{16 \cdot R^2} \right) \quad \text{...define } T(r).$$

To sketch the temperature profile in the cylinder, define a range variable r, varying from 0 to 0.1 m, with an increment of 0.005 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r)$, respectively. Click anywhere outside the graph region, and immediately the graph appears. See Fig. Ex. 5.19(b).

r := 0, 0.005, ..., 0.1 (*define a range variable r..starting value = 0, next value = 0.005 m and last value = 0.1 m*)

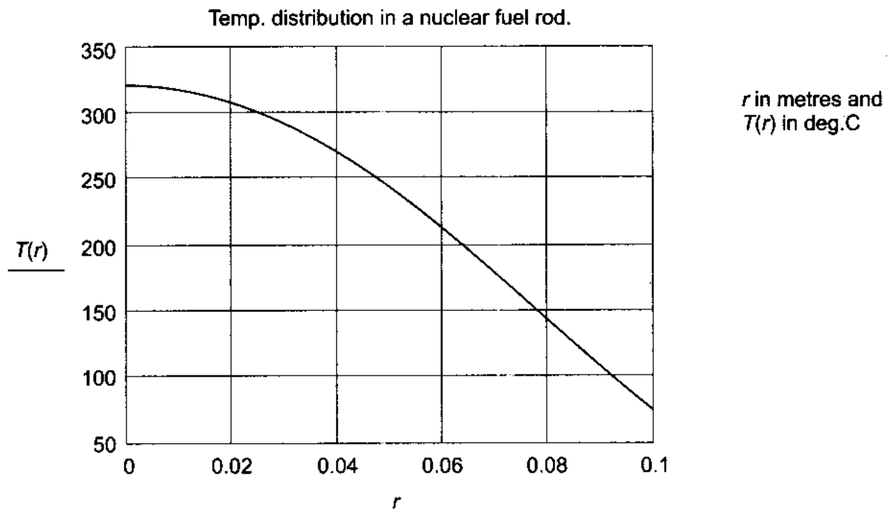

FIGURE Example 5.19(b)

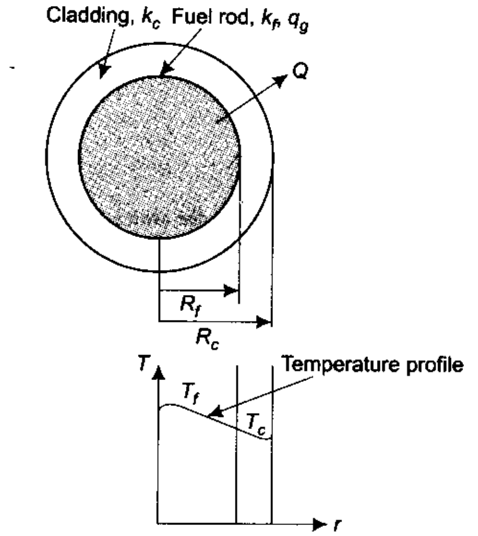

FIGURE 5.17 Cylindrical fuel rod with cladding

Note from the graph that temperature at the centre and surface are 321.1°C and 75°C, respectively.

5.5.4 *Heat Transfer in Nuclear Fuel Rod with Cladding*

Generally, fuel rod for use in a nuclear reactor is 'lagged' with a tight fitting cladding material, to prevent oxidation of the surface of the fuel rod by direct contact with the coolant. Usually, aluminium is used as the cladding material. We would like to analyse the temperature distribution and heat transfer in the combined system of (fuel rod + cladding). Remember that heat generation occurs only in the fissile material of the fuel rod and, in the cladding there is no heat generation.

In steady state, heat generated in the fuel rod is conducted through the cladding and then, dissipated to the coolant flowing around the cladding by convection. It is assumed that there is no contact resistance between the fuel rod and the cladding, i.e. there is continuity of heat flux and temperature profile at the interface. See Fig. 5.17.

Let R_f = outer radius of fissionable fuel rod
k_f = thermal conductivity of fuel rod
R_c = outer radius of cladding material
k_c = thermal conductivity of cladding material.

Let heat generation rate in the fuel rod vary with position according to the following relation:

$$q_g = q_o \cdot \left[1 - \left(\frac{r}{R}\right)^2\right]$$

where, q_o = heat generation rate per unit volume at the centre (i.e. at $r = 0$), and
R = outer radius of the solid fuel rod.

Assumptions:

(i) Steady state conduction
(ii) One-dimensional conduction, in the r direction only
(iii) Homogeneous, isotropic material with constant k
(iv) Internal heat generation at a varying rate: $q_g = q_o \cdot \{1-(r/R)^2\}$, (W/m^3).

For these assumptions, the controlling differential equation in cylindrical coordinates becomes:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\cdot\frac{dT}{dr} + \frac{q_g}{k} = 0 \quad \text{...(a)}$$

Multiplying by r:

$$r\cdot\frac{d^2T}{dr^2} + \frac{dT}{dr} + \frac{q_g \cdot r}{k} = 0$$

i.e.

$$\frac{d}{dr}\left(r\cdot\frac{dT}{dr}\right) + \frac{q_g \cdot r}{k} = 0 \quad \text{...(b)}$$

Now,

$$\frac{dT}{dr} = \frac{-q}{k}$$

(*from Fourier's law, where q is the heat flux*)

Therefore, $$\frac{d}{dr}\left(\frac{-r\cdot q}{k}\right)+\frac{q_g\cdot r}{k}=0$$

or, $$\frac{d}{dr}(r\cdot q)=q_g\cdot r$$ ((c) *since k is a constant.*)

Let us denote fuel rod and cladding materials by subscripts f and c, respectively.

Then,

for fuel rod:

$$\frac{d}{dr}(r\cdot q_f)=q_g\cdot r$$

i.e. $$\frac{d}{dr}(r\cdot q_f)=q_o\cdot\left[1-\left(\frac{r}{R_f}\right)^2\right]\cdot r$$ ...(d)

for cladding:

$$\frac{d}{dr}(r\cdot q_c)=0$$ ((e)...*since there is no heat generation in cladding*)

Integrating Eq. d $$r\cdot q_f=q_o\cdot\left(\frac{r^2}{2}-\frac{r^4}{4\cdot R_f^2}\right)+C_1$$

i.e. $$q_f=q_o\cdot\left(\frac{r}{2}-\frac{r^3}{4\cdot R_f^2}\right)+\frac{C_1}{r}$$ ...(f)

Integrating Eq. e $$r\cdot q_c=C_2$$

i.e. $$q_c=\frac{C_2}{r}$$ ...(g)

Now, apply the B.C.'s:

B.C. (i): q_f = finite, at $r = 0$

B.C.(ii): $q_f = q_c$, at $r = R_f$, i.e. at the interface

Then, from Eq. f and B.C.(i) $C_1 = 0$

And, from Eqs. f and g, and B.C.(ii):

$$\frac{C_2}{R_f}=q_f=q_o\cdot\left(\frac{R_f}{2}-\frac{R_f^3}{4\cdot R_f^2}\right)$$

i.e. $$\frac{C_2}{R_f}=\frac{q_o\cdot R_f}{4}$$

i.e. $$C_2=\frac{q_o\cdot R_f^2}{4}$$

Therefore, heat flux through the fuel rod and cladding may be written as:

$$q_f=-k_f\cdot\frac{dT_f}{dr}=q_o\cdot\left(\frac{r}{2}-\frac{r^3}{4\cdot R_f^2}\right)$$ ((h)...*from Eq. f*)

and, $$q_c = -k_c \cdot \frac{dT_c}{dr} = \frac{q_o \cdot R_f^2}{4 \cdot r}$$ ((i)...*putting value of* C_2 *in Eq.* g)

Next, temperatures T_f and T_c in fuel rod and cladding are obtained by integrating Eqs. h and i, respectively:

$$T_f = \frac{q_o}{k_f} \cdot \left(\frac{r^4}{16 \cdot R_f^2} - \frac{r^2}{4} \right) + C_3 \quad ...(j)$$

and, $$T_c = \frac{-q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln(r) + C_4 \quad ...(k)$$

Get C_3 and C_4 by applying the B.C.'s:

B.C.(iii): $T_c = T_w$, at $r = R_c$...i.e. at outer surface of cladding

B.C.(iv): $T_c = T_f$, at $r = R_f$, i.e. at the interface

Then, from Eq. k and B.C. (iii):

$$C_4 = T_w + \frac{q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln(R_c)$$

Immediately substituting C_4 in Eq. k, we get:

$$T_c = \frac{-q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln(r) + \left(T_w + \frac{q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln(R_c) \right)$$

i.e. $$T_c = T_w + \frac{q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln\left(\frac{R_c}{r} \right) \quad ...(5.55)$$

and, $$T_c - T_w = \frac{q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln\left(\frac{R_c}{r} \right) \quad ...(5.56)$$

Eq. 5.55 gives the temperature distribution in the cladding.

Eq. 5.56 gives the temperature drop across the cladding.

And, from Eq. j and B.C.(iv):

$$T_f = \frac{q_o}{k_f} \cdot \left(\frac{R_f^4}{16 \cdot R_f^2} - \frac{R_f^2}{4} \right) + C_3 = T_c$$

i.e. $$T_f = C_3 - \frac{3 \cdot q_o \cdot R_f^2}{16 \cdot k_f} = T_c \quad ...(l)$$

Then, from Eq. 5.55, and Eq. l:

$$C_3 = \frac{3 \cdot q_o \cdot R_f^2}{16 \cdot k_f} + T_w + \frac{q_o \cdot R_f^2}{4 \cdot k_c} \cdot \ln\left(\frac{R_c}{R_f} \right)$$

i.e. $$C_3 = T_w + \frac{q_o \cdot R_f^2}{4} \cdot \left(\frac{3}{4 \cdot k_f} + \frac{1}{k_c} \cdot \ln\left(\frac{R_c}{R_f} \right) \right)$$

Substituting C_3 in Eq. j:

i.e. $$T_f = \frac{q_o}{k_f}\cdot\left(\frac{r^4}{16\cdot R_f^2} - \frac{r^2}{4}\right) + C_3 \quad \text{....(j)}$$

we get:

$$T_f = \frac{q_o}{k_f}\cdot\left(\frac{r^4}{16\cdot R_f^2} - \frac{r^2}{4}\right) + T_w + \frac{q_o\cdot R_f^2}{4}\cdot\left(\frac{3}{4\cdot k_f} + \frac{1}{k_c}\cdot\ln\left(\frac{R_c}{R_f}\right)\right) \quad \text{...(5.57)}$$

Eq. 5.57 gives the temperature distribution in the fuel rod.

Maximum temperature in the fuel rod:

This occurs at the centre of the rod.

Putting $r = 0$ in Eq. 5.57, we get:

$$T_{max} = T_w + \frac{q_o\cdot R_f^2}{4}\cdot\left(\frac{3}{4\cdot k_f} + \frac{1}{k_c}\cdot\ln\left(\frac{R_c}{R_f}\right)\right) \quad \text{...(5.58)}$$

i.e. Eq. 5.58 gives the maximum temperature in the fuel rod.

5.6 Summary of Basic Conduction Relations, with Heat Generation

In this chapter, we have analysed steady state, one-dimensional heat transfer, with internal heat generation, in three important geometries, namely, plane slab, cylinder and sphere and derived relations for temperature distribution, maximum temperature difference and rate of heat transfer. We also studied the effect of variable thermal conductivity on some of these results. Since all these relations are practically important, they are tabulated in Table 5.1 to Table 5.7, for easy reference.

TABLE 5.1 Relations for steady state, one-dimensional conduction with internal heat generation, and constant k

Relation	Plane wall (both sides at T_w)	Plane wall (sides at T_1 and T_2)	Plane wall (insulated on one side)
Governing differential equation	$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0$	$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0$	$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0$
Temperature distribution	$T(x) = T_w + \frac{q_g}{2\cdot k}\cdot(L^2 - x^2)$	$T(x) = T_1 + \left[(L-x)\cdot\frac{q_g}{2\cdot k} + \frac{(T_2 - T_1)}{L}\right]$	$T(x) = T_w + \frac{q_g}{2\cdot k}(L^2 - x^2)$
Heat transfer rate at the surface, Q, (W)	$Q = q_g\cdot A\cdot L$	$Q_{left} = -k\cdot A\cdot\frac{dT(x)}{dx}$ at $x = 0$ $Q_{right} = -k\cdot A\cdot\frac{dT(x)}{dx}$ at $= x = L$	$q_g\cdot A\cdot L$
$T_{max} - T_w$, (C)	$\frac{q_g\cdot L^2}{2\cdot k}$	Equate $dT(x)/dx$ to zero; substitute resulting x in $T(x)$ to get T_{max}	$\frac{q_g\cdot L^2}{2\cdot k}$
Comments	L is **half-thickness** of slab; Maximum temperature occurs on the centre line	L is the thickness of slab	L is the thickness of slab; maximum temperature occurs on the insulated surface

TABLE 5.2 Relations for steady state, one-dimensional conduction with internal heat generation and k varying linearly with temperature as:

$k(T) = k_0(1 + \beta T)$

$k_m = k_0(1 + \beta T_m)$; $T_m = (T_1 + T_2)/2$

Relation	Plane wall of thickness, L (sides at T_1 and T_2)
Governing differential equation	$\frac{d}{dx}\left(k(T)\cdot\frac{dT}{dx}\right)+q_g=0$
Temperature distribution	$T(x)=\frac{-1}{\beta}+\sqrt{\left(\frac{1}{\beta}+T_1\right)^2-\frac{2\cdot x}{\beta\cdot L}\cdot(T_1-T_2)\cdot(1+\beta\cdot T_m)+\frac{q_g\cdot x}{\beta\cdot k_o}\cdot(L-x)}$
Heat transfer rate at the surface, Q, (W)	$Q_{left}=-k\cdot A\cdot\frac{dT(x)}{dx}$ at $x=0$ $Q_{right}=-k\cdot A\cdot\frac{dT(x)}{dx}$ at $x=L$
T_{max}, (C)	Equate $dT(x)/dx$ to zero; Subst. resulting x in $T(x)$ to get T_{max}

TABLE 5.3 Relations for steady state, one-dimensional conduction with internal heat generation, and constant k

Relation	Solid cylinder	Hollow cylinder (inside surface insulated)
Governing differential equation	$\frac{d^2T}{dr^2}+\frac{1}{r}\cdot\frac{dT}{dr}+\frac{q_g}{k}=0$	$\frac{d^2T}{dr^2}+\frac{1}{r}\cdot\frac{dT}{dr}+\frac{q_g}{k}=0$
Temperature distribution	$T(r)=T_w+\frac{q_g}{4\cdot k}\cdot(R^2-r^2)$	$T(r)=T_o+\frac{q_g\cdot r_i^2}{4\cdot k}\cdot\left[\left(\frac{r_o}{r_i}\right)^2-2\cdot\ln\left(\frac{r_o}{r}\right)-\left(\frac{r}{r_i}\right)^2\right]$
Heat transfer rate at the surface Q, (W)	$q_g\cdot\pi\cdot R^2\cdot L$	$q_g\cdot\pi\cdot\left(r_o^2-r_i^2\right)\cdot L$
$T_{max}-T_w$, (C)	$\frac{q_g\cdot R^2}{4\cdot k}$	$\frac{q_g\cdot r_i^2}{4\cdot k}\cdot\left\vert\left(\frac{r_o}{r_i}\right)^2-2\cdot\ln\left(\frac{r_o}{r_i}\right)-1\right\vert$
Comments	L is length of cylinder; maximum temperature occurs at the centre	L is length of cylinder; maximum temperature occurs on the inside surface

5.7 Summary

In this chapter, we studied one-dimensional, steady state heat transfer through simple geometries of a plane slab, cylinder (both solid and hollow) and sphere, with internal heat generation. Whether the heat generation rate is uniform or varying with position, the solution technique is, always, to start with the appropriate general differential equation and solve it by applying the boundary conditions. Once the temperature distribution is known, rate of heat transfer at any location is easily calculated by applying Fourier's law.

Problems of heat transfer when the thermal conductivity varies with temperature were also studied.

Applications of these techniques to some practical cases with internal heat generation, such as dielectric heating, current carrying conductor, nuclear fuel rods with and without cladding, etc. were discussed.

TABLE 5.4 Relations for steady state, one-dimensional conduction with internal heat generation, and constant k

Relation	Hollow cylinder (outside surface insulated)	Hollow cylinder (surfaces at T_1 and T_2)
Governing differential equation	$\frac{d^2T}{dr^2}+\frac{1}{r}\cdot\frac{dT}{dr}+\frac{q_g}{k}=0$	$\frac{d^2T}{dr^2}+\frac{1}{r}\cdot\frac{dT}{dr}+\frac{q_g}{k}=0$
Temperature distribution	$T(r)=T_i+\frac{q_g\cdot r_o^2}{4\cdot k}\cdot\left[2\cdot\ln\left(\frac{r}{r_i}\right)+\left(\frac{r_i}{r_o}\right)^2-\left(\frac{r}{r_o}\right)^2\right]$	$\frac{T(r)-T_i}{T_o-T_i}=\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}+\frac{q_g}{4\cdot k}\cdot\frac{(r_o^2-r_i^2)}{(T_o-T_i)}\cdot\left[\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}-\frac{\left(\frac{r}{r_i}\right)^2-1}{\left(\frac{r_o}{r_i}\right)^2-1}\right]$
Heat transfer rate at the surface, Q, (W)	$q_g\cdot\pi\cdot(r_o^2-r_i^2)\cdot L$	$Q_{inner}=-k\cdot A_i\cdot\frac{dT(r)}{dr}$ at $r=r_i$ $Q_{outer}=-k\cdot A_o\cdot\frac{dT(r)}{dr}$ at $r=r_o$
$T_{max}-T_w$, (C)	$\frac{q_g\cdot r_o^2}{4\cdot k}\cdot\left\vert 2\cdot\ln\left(\frac{r_o}{r_i}\right)+\left(\frac{r_i}{r_o}\right)^2-1\right\vert$	$\frac{q_g\cdot r_i^2}{4\cdot k}\cdot\left\vert\left(\frac{r_o}{r_i}\right)^2-2\cdot\ln\left(\frac{r_o}{r_i}\right)-1\right\vert$
Comments	L is length of cylinder; maximum temperature occurs on the outside surface.	L is length of cylinder; Position of maximum temperature is given by: $r_m=\sqrt{\frac{q_g\cdot(r_o^2-r_i^2)-4\cdot k\cdot(T_i-T_o)}{q_g\cdot 2\cdot\ln\left(\frac{r_o}{r_i}\right)}}$

TABLE 5.5 Relations for steady state, one-dimensional conduction with internal heat generation and k varying linearly with temperature as:

$$k(T)=k_o(1+\beta T)$$

$$k_m=k_o(1+\beta T_m);\ T_m=(T_1+T_2)/2$$

Geometry	Temperature distribution, $T(r)$
Solid cylinder	$T(r)=\frac{-1}{\beta}+\sqrt{\left(\frac{1}{\beta}+T_w\right)^2+\frac{q_g\cdot(R^2-r^2)}{2\beta\cdot k_o}}$
Hollow cylinder with inside surface insulated	$T(r)=\frac{-1}{\beta}+\sqrt{\left(\frac{1}{\beta}+T_i\right)^2-\frac{q_g\cdot r_i^2}{2\cdot\beta\cdot k_o}\cdot\left[\left(\frac{r}{r_i}\right)^2-2\cdot\ln\left(\frac{r}{r_i}\right)-1\right]}$
Hollow cylinder with outside surface insulated	$T(r)=\frac{-1}{\beta}+\sqrt{\left(\frac{1}{\beta}+T_o\right)^2-\frac{q_g\cdot r_o^2}{2\cdot\beta\cdot k_o}\cdot\left[2\cdot\ln\left(\frac{r_o}{r}\right)-\left(\frac{r_o}{r}\right)^2-1\right]}$

TABLE 5.6 Relations for steady state, one-dimensional conduction with internal heat generation, and constant k

Relation	Solid sphere
Governing differential equation	$\frac{d^2T}{dr^2}+\frac{2}{r}\cdot\frac{dT}{dr}+\frac{q_g}{k}=0$
Temperature distribution	$T(r)=T_w+\frac{q_g}{6\cdot k}(R^2-r^2)$
Heat transfer rate at the surface, Q, (W)	$\frac{4}{3}\cdot\pi\cdot R^3\cdot q_g$
$T_{max}-T_w$, (C)	$\frac{q_g\cdot R^2}{6\cdot k}$
Comments	Maximum temperature occurs at the centre.

TABLE 5.7 Relations for steady state, one-dimensional conduction with internal heat generation and k varying linearly with temperature as:

$$k(T)=k_o(1+\beta T)$$
$$k_m=k_o\,(1+\beta T_m)\,;\ T_m=(T_1+T_2)/2$$

Relation	Solid sphere
Governing differential equation	$\frac{d^2T}{dr^2}+\frac{2}{r}\cdot\frac{dT}{dr}+\frac{q_g}{k}=0$
Temperature distribution	$T(r)=\frac{-1}{\beta}+\sqrt{\left(\frac{1}{\beta}+T_w\right)^2+\frac{q_g\cdot(R^2-r^2)}{3\beta\cdot k_o}}$
Heat transfer rate at the surface, Q, (W)	$\frac{4}{3}\cdot\pi\cdot R^3\cdot q_g$
Comments	Maximum temperature occurs at the centre.

Several problems were solved and graphical representation of temperature distribution using Mathcad was highlighted.

Finally, at the end of the chapter, the basic relations developed in this chapter for the aforesaid three geometries, are tabulated for easy reference.

In the next chapter, we will study an important application of combined heat transfer of conduction and convection, namely fins or extended surfaces.

Questions

1. Why are the cases with heat generation analysed? Give some practical examples.
2. Derive an expression for temperature distribution under one-dimensional steady state heat conduction with heat generation of q_g (W/m^3) for the following system:
   Plate of wall thickness L, thermal conductivity k, temperature being T_1 and T_2 at the two faces.
3. Pressure vessel for a nuclear reactor is approximated as a large flat plate of thickness L. Inside surface at $x = 0$ is insulated. Outside surface at $x = L$ is maintained at a uniform temperature T_2. Gamma ray heating of the plate is represented by:

   $$q_g\,(x)=q_o\exp(-ax),\ (\text{W/m}^3)\text{ where } q_g,\ q_o \text{ and } a \text{ are constants.}$$

   (a) Develop an expression for temperature distribution in the plate.
   (b) Develop an expression for temperature at the insulated surface ($x = 0$)
   (c) Develop an expression for the heat flux at the outer surface, i.e. at $x = L$.

4. In the above problem, if the surface at $x = 0$ is insulated and the surface at $x = L$ dissipates heat by convection with a convective heat transfer coefficient of h to a fluid at a temperature of T_a, develop an expression for the temperature distribution in the wall and the temperature of the insulated surface.
5. For solid cylinder of radius R, with heat generation q_g (W/m^3), surface temperature T_w and centre temperature T_0, show that temperature distribution is given by:

$$(T - T_w)/ (T_0 - T_w) = 1 - (r/R)^2$$

6. Determine the one-dimensional temperature distribution $T(r)$ for a solid cylinder of radius R, constant thermal conductivity k, when the heat generation rate varies as:
   $q_g(x) = q_o\{1 - (r/R)\}$, (W/m^3) where q_g, q_o are constants. Boundary surface at $r = R$ is kept at zero deg.C.
7. A hollow cylinder of inside radius r_i, outside radius r_o, has its inner and outer surfaces maintained at uniform temperature T_1 and T_2. Inside surface is insulated. Thermal conductivity k is constant and there is uniform heat generation rate q_g (W/m3). Show that:

$$T_{max} - T_2 = \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i} \right)^2 - 2 \cdot \ln\left(\frac{r_o}{r_i} \right) - 1 \right]$$

8. Derive an expression for the variation of temperature along the radius for a solid sphere of constant k when there is uniform heat generation in the solid. Temperature of the surface ($r = R$) is T_w.
9. How does the temperature distribution change if the thermal conductivity varies linearly with temperature as: : $k(T) = k_o(1 + \beta T)$, where k_o and β are constants.
10. In a solid sphere of radius R, heat is generated at a rate of $q_g = q_o\{1 - (r/R)^2\}$, W/m^3, where q_o is a constant. Boundary surface at $r = R$ is maintained at a constant temperature T_w. Develop an expression for the steady state temperature distribution, $T(r)$.

Problems

Plane slab:

1. A plane wall 6 cm thick generates heat internally at the rate of 0.30 MW/m^3. One side of the wall is insulated, and the other is exposed to an environment at 93°C. The convection heat transfer coefficient between the wall and the environment is 570 W/m^{2}K. Thermal conductivity of the wall is k = 21 W/(mK). Calculate the maximum temperature in the wall.
2. A large, 3 cm thick plate (k = 18 W/(mK)) has a uniform heat generation rate of 5 MW/m^3. Both the sides of the plate are exposed to an ambient at 25°C. Find out the maximum temperature in the plate and where it occurs. Draw the temperature profile in the plate.
3. A 4 cm thick brass plate (k = 110 W/(mC)), has uniform internal heat generation rate of 2×10^5 W/m^3. Its one face is insulated and the other face is exposed to a stream of cooling air at 20°C flowing with a heat transfer coefficient of 45 W/(m^{2}C). Find the maximum temperature in the plate and where it occurs. Draw the temperature profile.
4. A steel plate 25 mm thick, (k = 50 W/(mK)) has uniform volumetric heat generation rate of 50 MW/m^3. Its two surfaces are maintained at 150°C and 100°C. Neglecting end effects, determine:
   (i) position and value of maximum temperature
   (ii) heat flow rate from each surface.

Cylinder:

5. A S.S. rod of 2 cm diameter carries an electric current of 900 A. Thermal and electrical conductivities of the rod are 16 W/(mC) and 1.5×10^4 (Ohm cm)$^{-1}$, respectively. What is the temperature difference between the centre line and periphery in steady state?
6. A copper wire 1 mm in diameter is insulated with a plastic to an outer diameter of 3 mm and is exposed to an environment at 40°C. Find the maximum current carried by the wire in amperes without heating any point of plastic above 90°C. Heat transfer coefficient from the outer surface of the plastic to the surrounding is 10 W/(m^{2}K), k of plastic = 0.4 W/(mK), electrical conductivity of copper is 5×10^7 ohm^{-1} m^{-1}. Also, find the maximum temperature of the wire. Given: k of copper = 380 W/(mK).
7. An electric cable of k = 20 W/(mC), 3 mm in diameter and 1 m long, has resistivity ρ = 70 ohm.cm. A current of 190 A flows through it and the wire is submerged in a fluid at a temperature of 90°C with a heat transfer coefficient of 4000 W/(m^{2}C). Calculate the centre temperature of the wire.
8. A chemical reaction takes place in a packed bed (k = 0.5 W/(mC)) between two coaxial cylinders of radii 10 cm and 35 cm. The inner surface is insulated and is maintained at 500°C. If the reaction produces a uniform heat generation of 500 kW/m^3, find the temperature of the outer surface.

9. An electric resistance wire of radius 1.5 mm has k = 25 W/(mC). It is heated by passing a current and heat generation rate is 2×10^9 W/m^3. Determine the difference between the temperature at the centre line and the surface if the surface is maintained at a constant temperature.

10. Consider a copper rod of radius 6 mm, k = 380 W/(mK) wherein heat is generated at a uniform rate of 4.5×10^8 W/m^3. It is cooled by convection from its surface to ambient air at 30°C with h = 1800 W/(m^{2}C). Determine the surface temperature of the rod.

11. A hollow S.S. tube with r_i = 25 mm, r_o = 35 mm, k = 15 W/(mK), electrical resistivity $\rho = 0.7 \times 10^{-6}$ Ohm.m, has uniform heat generation inside it, induced by an electric current. The heat is transferred by convection to air flowing through the tube. If the air temperature is 400 K and the convective heat transfer coefficient is 150 W/(m^{2}K) and the maximum allowable temperature anywhere in the tube is 1400 K, determine the maximum allowable electric current. Assume that the tube surface is perfectly insulated. Draw the temperature profile in the cylindrical shell.

12. A cylindrical rod, 6 cm radius, generates heat at a rate of 2.5 MW/m^3. k of the material is 20 W/(mK). It is clad with a stainless steel layer of 6 mm thickness (k = 14 W/(mK), whose outer surface is cooled by a fluid at 180°C with a heat transfer coefficient of 600 W/(m^{2}K). Determine the temperature at the centre of the rod and also on the outer surface and interface. Draw the temperature profile in both the rod and the cladding.

13. Rate of heat generation in a cylindrical fuel rod is given by:

$$q_g = q_o[1 - (r/R)^2], \text{ W/m}^3, \text{ where } R \text{ is the radius of the fuel rod.}$$

   (a) Calculate the temperature drop from the centre line to the surface of the rod, for the following data: diameter of fuel rod = 25 mm, $q_o = 80 \times 10^6$ W/m^3, k = 20 W/(mK).

   (b) If the heat removal rate from the outer surface of the rod is 0.2 MW/m^2, what would be the temperature drop from the centre to the surface?

Sphere:

14. A homogeneous sphere of 9 cm diameter has a uniform heat generation rate of 5×10^7 W/m^3. k of the material is 15 W/(mK). If the surface temperature is maintained at 75°C,
   (i) determine the temperature at the centre of the sphere
   (ii) draw the temperature profile along the radius.

   Assume steady state, one dimensional conduction.

15. A solid sphere of radius R = 6 mm, k = 25 W/(mC), has a uniform heat generation rate of 2500 W/m^3. Heat is carried away by convection at its outer surface to ambient air at 30°C with a heat transfer coefficient of 25 W/(m^{2}C). Determine the steady state temperature at the centre and outer surface of the sphere.

16. Average heat generation during ripening of oranges is estimated as 325 W/m^3. Assuming the orange to be a sphere of diameter 10 cm, and k = 0.15 W/(mC), find out the centre temperature of the orange if the surface is maintained at 10°C. Draw the temperature profile along the radius.

17. A hollow sphere of 10 cm ID, 20 cm OD, is made of a material of k = 18 W/(mK). Heat is generated internally at a uniform rate of 3 MW/m^3. Inside surface of the sphere is insulated. Develop an expression for the temperature profile in the sphere and determine the maximum temperature in the material, if the outside surface temperature is maintained at 300°C. Draw the temperature profile in the shell.